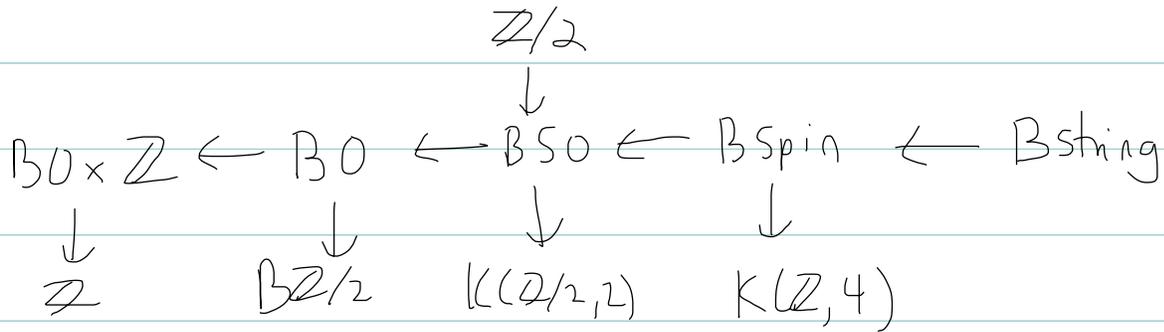


Eric Peterson - 4/25/13

• Some interesting spaces are given in terms of Postnikov towers

"Morava K-theorys of Eilenberg MacLane spaces"



Spectral sequence

Inputs

Serre SS

Rothenberg - Steenrod

Eilenberg - Moore

$H_*^{cell}$

$X\langle q \rangle \otimes K_*$

Notation

$K(n)_* \underline{H}G_q$

$K_* X\langle q \rangle$

$X_q = q^{\text{th}}$  space  
in  $\Omega$ -spectrum

Output:  $K_* X\langle q+1 \rangle$

Simple case:  $G$  finite cyclic grp  $(p^h)$   $q=1$

(1)  $K_* = \mathbb{F}_p[V_n^{\pm 1}] \quad |V_n| = 2(p^h - 1)$

(2)  $[p](x) = xp^h$

$$\begin{array}{ccccccc}
 \underline{HZ}_0 & \rightarrow & \underline{HZ}_0 & \rightarrow & \underline{HZ/p^k}_0 & \rightarrow & \underline{HZ}_1 \rightarrow \underline{HZ}_1 \rightarrow \underline{HZ/p^k}_1 \rightarrow \underline{HZ}_2 \\
 \parallel & & \parallel \\
 \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}/p^k & & \mathbb{B}\mathbb{Z} & \mathbb{B}\mathbb{Z} & \mathbb{B}\mathbb{Z}/p^k & & \mathbb{C}P^\infty
 \end{array}$$

Gysin  $K_* \mathbb{B}\mathbb{Z}/p^k \rightarrow K_* (\mathbb{C}P^\infty)$

$\swarrow \partial$   $\searrow -ne = \mathbb{Z}$   
 $K_* (\mathbb{C}P^\infty)$

$$\begin{array}{ccc}
 S_1 = S_1 & & \\
 \downarrow & & \downarrow \\
 \mathbb{B}\mathbb{Z}/p^k \rightarrow * & \Rightarrow & \mathbb{Z} = -n \times p^{kn} \quad K_* \mathbb{C}P^\infty = \\
 \downarrow & & \downarrow & & K_* ([x]) \\
 \mathbb{C}P^\infty \xrightarrow{[p^k]} \mathbb{C}P^\infty & & & & 
 \end{array}$$

$$\langle \sigma \cap \omega, \omega' \rangle = \langle \sigma, \omega \cup \omega' \rangle$$

$$K_* \mathbb{C}P^\infty = K_* \{B_0, B_1, \dots\}$$

$$(B_i, X^j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$\langle B_i \cap X^{pk}, X^j \rangle = \langle B_i, X^{pk+j} \rangle$$

$$\Rightarrow \bar{\Phi}(B_i) = B_i - p^n K \quad \Rightarrow \bar{\Phi} \text{ surjective}$$

$$K_*(B\mathbb{Z}/p^n) \hookrightarrow K_* \mathbb{C}P^\infty$$

$$\Rightarrow \begin{array}{ccc} & \uparrow \circ & \swarrow \\ & K_*(\mathbb{C}P^\infty) & \end{array}$$

$$K^* B\mathbb{Z}/p^k \leftarrow K^* \mathbb{C}P^\infty$$

$$\begin{array}{ccc} \circ \downarrow & \nearrow & \mathbb{C}P^k \langle X \rangle \\ & K^* \mathbb{C}P^\infty & \end{array}$$

$$\begin{aligned}
K_* B\mathbb{Z}/p^k &= K_* (\mathbb{C}P^\infty / (\mathbb{C}P^{n_k})(x)) \\
&= K_* [\mathbb{C}P^\infty] / (\mathbb{C}P^{n_k})(x) \\
&= K_* [\mathbb{C}P^\infty] / (x^{p^{n_k}}) \\
&= p^k \text{ torsion in formal group}
\end{aligned}$$

$$X_k = \text{Spt } K^0 X$$

$$B\mathbb{Z}/p^k \simeq \mathbb{C}P^\infty / (\mathbb{C}P^{n_k})$$

Finishes  $q=1$  computation:

The computation for  $q > 1$ :

(a) Hopf ring: The  $E-M$  spaces are...

$$\text{Spaces} \rightsquigarrow \underline{H\mathbb{Z}/p^k}_q \xrightarrow{\Delta} \underline{H\mathbb{Z}/p^k}_q \times \underline{H\mathbb{Z}/p^k}_q$$

Get comult on  $K_*$

$$\text{Loop spaces} \rightsquigarrow \underline{H\mathbb{Z}/p^k}_q \times \underline{H\mathbb{Z}/p^k}_q \xrightarrow{*} \underline{H\mathbb{Z}/p^k}_q$$

$$\begin{aligned} \Omega^\infty \text{ of ring spectrum} \rightsquigarrow & \underline{H\mathbb{Z}/p^k}_q \times \\ & \underline{H\mathbb{Z}/p^k}_q \xrightarrow{\circ} \\ & \underline{H\mathbb{Z}/p^k}_{q+q'} \end{aligned}$$

$$\text{Distributivity: } X(y+z) = Xy + Xz$$

$$X \xrightarrow{\circ} X \times X$$

$$\Delta X = \sum X' \otimes X'' \quad \text{w/alg classes}$$

$$X \circ (Y * Z) = \sum (X' \circ Y) * (X'' \circ Z)$$

"Hopf ring distributivity"

Ring obj in cat of  $K_*$  coalg

b)  $H\mathbb{Z}/p^k$  is a connective spectrum

$$\underline{H\mathbb{Z}/p^k}_{q+1} = | \beta(\underline{H\mathbb{Z}/p^k}_q) |$$

Skeletal filtration:

$$E_{*,*}^2 = \text{Tor}_{*,*}^{K_* H\mathbb{Z}/p^k} (K_*, K_*)$$

$$\Rightarrow K_* \underline{H\mathbb{Z}/p^k}_{q+1}$$

c) These fit together

i) This is a SS of Hopf algebras

ii) This induces a  $\circ$  map on SS

$$d(x \circ y) = x \circ d(y)$$

$\uparrow$

earlier  
spectral  
sequence

(probably permanent  
cycle)

These classes  
live in  
different  
spectral  
sequences

$$x \in K_* \underline{H\mathbb{Z}/p^k}$$

$$y \in E^r_{*,*} \text{ BSS}$$

Attempt an induction

a) Base case:  $\text{Tor}_{*,*}^{K_* \underline{H\mathbb{Z}/p^k}} \Rightarrow K_* \underline{H\mathbb{Z}/p^k}$

b) Produce a decomposition formula for lots of classes.

$$\begin{aligned}
 a) \quad K_* \underline{\text{HZ}/p}_0 &= K_* \{ [0], \dots, [p^k - 1] \} \\
 &= K_*^{[0] - [0]} \left( [1] - [0] \right)^{p^k}
 \end{aligned}$$

$$\Rightarrow K_* \underline{\text{HZ}/p}_0 = K_* [y] / y^{p^k}$$

$$\begin{aligned}
 \text{Tot} \quad K_* [y] / y^{p^n} &= \wedge (\nabla_{a_\phi}) \otimes \\
 * \quad | \quad * & \\
 &\Rightarrow \Gamma (\phi_{a_\phi}) \\
 &\text{divided} \\
 &\text{power}
 \end{aligned}$$

$$d(\phi_{a_\phi})^{[p^{nk}]} = \nabla a_\phi$$

$$d(\phi_{a_\phi})^{[p^{nk+i}]} = \nabla a_\phi \circ (\phi_{a_\phi})^{[i]}$$

Thm: (Ravanel-Wilson)

$$i) \quad K_* \underline{H\mathbb{Z}/p^k}_q = \underbrace{\text{Alt}^q}_{\substack{\uparrow \\ \text{exterior} \\ \text{power}}} K_* \underline{H\mathbb{Z}/p^k}_1$$

under the  $\circ$ -product

$$ii) \quad \text{Tor}_{*,*}^{K_* \underline{H\mathbb{Z}/p^k}_q} = \bigotimes_{\substack{I \text{ (length} \\ q \\ \text{multi-} \\ \text{index)}}} (\wedge^{[I]} \nabla a_I \otimes \wedge^{[I]} \phi a_I)$$

$$iii) \quad (\phi_{a_I})^{[p^j]} = (\phi_{a_{[I_1, \dots, I_{q-1}]}})^{[p^j]} \circ a_{[I_q + j]}$$

modulo  $*$  - decomposables

"Rewriting formula"

iv) This determines the differentials and the multiplicative extension problems in BSS using the previous one

v) This kills just enough to have i) for  $q+1$ .

① For  $q > n$ , this Hopf alg is just  $K_*$

② Spf  $\frac{K_* \mathbb{H}\mathbb{Z}/p^\infty}{q}$  is a  $p$ -divis group

$$K^* \underline{HZ/p^\infty}_q = K^* \langle x_1, \dots, x_{\binom{n-1}{q-1}} \rangle$$

[ dim'l:  $q=1, n$

$$\text{ht} : K^* \underline{HZ/p^\infty}_q = 1$$

when  $n$  is even  $\cong \mathbb{G}_m$

odd, becomes  $\cong$  after adjoining  $\zeta_{2p-2}$

Also gives answer for Morava E-theory

(3) If you replace  $K$  with  $E$   
everything works.

$$(4) \quad B\mathbb{Z}/p^k_E = (\mathbb{C}P^\infty_E \times \mathbb{C}P^k) \quad (\text{Hopkins})$$

(Kuhn  
Ravard)

$$(5) \quad E_* \underline{H\mathbb{Z}/p^k}_a = \text{Alt}^a E_* \underline{H\mathbb{Z}/p}_1 \quad (p.)$$

Exercise use this method

$$H\mathbb{F}_p \rightarrow \underline{H\mathbb{F}_p} \quad p=2$$