

Thm: (Nishida)

Every $\alpha \in \pi_n(S^0)$ with $n > 0$ is nilpotent.

Def: Suppose E is an associative ring spectrum

$$f: X \rightarrow Y \text{ s.t. } id_E \wedge f^{nn} : E \wedge X^{nn} \rightarrow E \wedge Y^{nn}$$

null-hpc for some n , then say f is E -nilpotent

If $f: X \rightarrow X$, say f is " E -composition" nilpotent if

$id_E \wedge f^{nn}$ is null-homotopic.

If R is a ring spectrum, then say $\alpha \in \pi_* R$ is E -Murewicz nilpotent if for any bracketing

$$\mu^n: R^n \rightarrow R$$

i.e. nilpotent in ring MUR

We have $\mu_E \wedge (\mu^n \alpha^n)$

If $\alpha \in \pi_n(R)$, then can form a self map

$$\bar{\alpha}: \Sigma^n R \rightarrow R$$

$$S^n \wedge R \rightarrow R \wedge R \xrightarrow{\mu} R$$

$$\bar{\alpha}^{-1}R = \text{Tel}(R, \alpha) = \underset{\bar{\alpha}}{\text{holim}} (R \rightarrow \bar{\Sigma}^n R \rightarrow \Sigma^{-2n} R \rightarrow \dots)$$

$$\text{Tel}_d(R, \alpha) \longrightarrow \text{Tel}_{(d+1)}(R, \alpha)$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \Sigma^{-dn} R & \xrightarrow{\bar{\alpha}} & \Sigma^{-d(n+1)} R \end{array}$$

Lemma: $\bar{\alpha}^{-1}R \wedge E \cong *$ \Leftrightarrow α is E -Murewicz nilpotent

Nilpotence Theorem

I - Suppose R is a connective, associative ring spectrum of finite type. Then

$\alpha \in \pi_n(R)$ is MW-Murewicz nilpotent

$\Rightarrow \alpha$ nilpotent

II - R any ring spectrum

III If $f: F \rightarrow X$ and F finite spectrum

Then f is MU - Λ nilpotent $\Rightarrow f$ nilpotent.

IV - If we have

$$\rightarrow X_{n+1} \rightarrow X_n \rightarrow X_{n-1} \rightarrow \dots$$

and X_n is C_n connective and $C_n \geq mn + b$

with $MU(f_n) = 0$ for all n , then $\text{Tel}(X_n, f_n) \simeq *$

V If R is p -local and $\alpha \in \pi_*(R)$ is BP-Murewicz, then α is nilpotent

VI α $K(n)$ -Murewicz nilpotent $\forall 0 \leq n < \infty$.

Cor - Nishida

$\alpha \in \pi_n S^0$ then it is torsion, so MU -image is 0.

Prop: I \Rightarrow III

$\alpha: F \rightarrow X$ is MU-smash nilpotent.

$\alpha: MU \wedge F \rightarrow MU \wedge X$ is zero

$\bar{\alpha}: S^0 \rightarrow X \wedge DF$ is 0 after $\wedge MU$

S^0 and $I_+ \wedge S^0$ are small so take some finite

Subspectrum of $X \wedge DF$

s.t.

α and $MU \wedge \alpha$ factor through X'

Suspend so X' connected

$T(X') = \bigvee_n (X')^{\wedge n}$ satisfies the conditions
of 1

$X' \rightarrow T(X')$

$i_n \bar{\alpha}$ is MU-Murewicz nilpotent in $T(X')$
so $\bar{\alpha}$ is nilpotent.

III \Rightarrow II

Bousfield
class of
 \downarrow

IV \Rightarrow VI : $\langle Bp \rangle = \langle K(0) \rangle v \dots v \langle K(n) \rangle v \langle P(n+1) \rangle$

$$P(n+1) = \mathbb{Z}/p [v_{n+1}, v_{n+2}, \dots]$$

Ravanel's nilpotence conjecture

$$f: \sum^d X \xrightarrow{f} X \quad w/ \quad MU_*(f) = 0$$

X finite then f is composition nilpotent.

$$\text{Pf: } \text{Tel}(X, f) \cong *$$

So some composite of f is 0.

Outline of proof of I

$$* \cong \Omega \text{SU}(1) \rightarrow \Omega \text{SU}(2) \rightarrow \dots \rightarrow \Omega \text{SU} \xrightarrow{B} \text{BU}$$

$$\text{BU} \cong \text{BU} \times 0 \subset \text{BU} \times \mathbb{Z}$$

$$X(n) = \Omega \text{SU}(n) \quad \text{- Taut bundle pulled back}$$

$$S^0 = X(0) \rightarrow X(1) \rightarrow \dots \rightarrow X(\infty) = \text{MU}$$

↑
Commutative
Ring spectrum

$X(n) \rightarrow MU$ is $2n-1$ connected

$\alpha \in \pi_d(\mathbb{R})$ with zero MU-Hurewicz image

then α has $\cap X(n)$ -Hurewicz image for large enough n ,

Thm. If α is $X(n+1)$ -Hurewicz nilpotent,

then it is $X(n)$ -Hurewicz nilpotent

$$X(n+1) \wedge \alpha^{-1} \mathbb{R} \simeq^* \Rightarrow X(n) \wedge \alpha^{-1} \simeq^*$$

actually do p -locally.

Homology of Various Spaces

$$\text{Fix } L_0 \in \mathbb{C}P^{n-1}$$

$$\Sigma \mathbb{C}P^{n-1} \rightarrow SU(n)$$

$$S^1 \times \mathbb{C}P^{n-1} \rightarrow SU(n)$$

$$(z, L) \longmapsto (z^{\pi} \pi_{L_0} + \pi_{L_0^\perp}) (z^{\pi} \pi_L + \pi_{L^\perp})$$

$$\mathbb{C}P^{n-1} \rightarrow \Omega SU(n)$$

$$\downarrow \qquad \downarrow$$

$$\mathbb{C}P^n \rightarrow \Omega SU(n+1)$$

$$\downarrow \qquad \downarrow$$

$$\mathbb{C}P^\infty \rightarrow \Omega SU \xrightarrow{\beta^{-1}} BU$$

↙ Taut line bundle

virtual v.b. over $\mathbb{C}P^n$ is $\mathcal{O}(-1) - 1$

$$(\mathbb{C}P^{n-1})^V \cong \Sigma^{-2}(\mathbb{C}P^n)^V$$

... $\Sigma^{-2} \mathbb{C}P^\infty \rightarrow MU$ complex orientation

$$H_* (\mathbb{C}P^{n-1}) = \{ \beta_0, \dots, \beta_n \} \text{ dual to } \chi_i$$

$$H_* (\Omega SU(n)) = \mathbb{Z}[\beta_0, \dots, \beta_n] / (\beta_0 - 1)$$

Thom iso gives $H_* (X(n)) = \mathbb{Z}[\beta_0, \dots, \beta_n] / (\beta_0 - 1)$

β_i comes from $\Sigma^{-2} \mathbb{C}P^{n+1}$

Prop: Suppose $k \leq n$

$$X(n)(\mathbb{C}P^k) = X(n)_* \{ \beta_1, \dots, \beta_k \}$$

$$X(n) \underset{*}{X}(k) = (b_0, \dots, b_k) / (b_0 - 1)$$

Thm: $X(n) \underset{*}{X}(n)$ is flat over $X(n) \underset{*}$

More reductions

$$\begin{array}{ccc} F'_{*,n} & \longrightarrow & \Omega \text{SU}(n+1) \\ \downarrow & & \downarrow \\ J_k S^{2n} & \longrightarrow & \Omega S^{2n+1} \end{array}$$

$F_{k,n}$ - Thom spectrum

$$F_{0,n} = X(n) \rightarrow \dots \rightarrow F_{k,n} \rightarrow \dots \rightarrow X(n+1)$$

$$G_{K,n} = (F_{p^{K-1},n})_{(p)}$$

Thm: If $X^{(n+1)} \wedge X^{-1}R \simeq *$ then
 $G_{K,n} \wedge \alpha^{-1}R \simeq *$ for large enough K

Thm $\langle G_{K,n} \rangle \simeq \langle G_{K+1,n} \rangle$ ↙ Bousfield class

$$G_{0,n} \wedge \alpha^{-1}R \simeq *$$

$$X^{(n)}_{(p)} \wedge \alpha^{-1}R \simeq *$$