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Thm (Goerss - Henn - Mahowald
- Rezk) ($p=3$)

"Resolutions of the $K(2)$ -
local Sphere"

There is a sequence of maps b/w Spectra

$$\begin{aligned} L_{K(2)} K^{\circ} &\rightarrow E_2^{hG_{24}} \rightarrow \sum^8 E_2^{hSD_{16}} \vee \sum_2^{hG_{24}} \\ &\rightarrow \sum^8 E_2^{hSD_{16}} \vee \sum^{40} E_2^{hS_{616}} \rightarrow \sum^{40} E_2^{hSD_{16}} \vee \sum^{48} E_2^{hG_{24}} \\ &\rightarrow \sum^{48} E_2^{hG_{24}} \end{aligned}$$

Such that the composition of any two maps is null and all possible Toda brackets are zero modulo indeterminacy

① Toda brackets = 0 \Rightarrow This refines to a tower of fibrations with $L_{K(2)} S^{\circ}$ at the top

• Not an E_2 Adams resolution. In particular, $E_2^{hG_{24}}$ are not E_2 -injective

Preliminaries At height $n=p-1$, G_n has infinite cohomological dimension. But it has finite index subgroups

with finite coh dim i.e. finite virtual cohomological dimension.

Def: A Morava module is a complete $(E_n)_*$ module w/ action of G_n such that these actions commute.

Prototypical example: $E_*^v X = \pi_* L_{K(n)}^{(E_n X)}$

$= E_* X$ (at least if $K(n)_* X$ is

in even degree)

Let M be a Morava module.

Consider $M \underset{\text{cont}}{\sqsubset} (G_n M)$

is a Morava module

The E_* action

$$(a\varphi)(x) = a \varphi(x)$$

G_n action

$$(g\varphi)(x) = g \varphi g^{-1} x$$

Thm: (Morava) $E_* E = \text{Hom}^e(G_n, E_*)$

Thm (D-H): $E_* E_n^{hK} \cong \text{Hom}^e(G_n/K, E_*)$

K finite

$M \uparrow_{K}^{G_n}$ denote the induced module of

$\mathbb{Z}_p(K)$ module

$$M \uparrow_{K}^{G_n} = \mathbb{Z}_p((G_n)) \hat{\otimes}_{\mathbb{Z}_p(K)} M$$

If $M = \mathbb{Z}_p$, we have $\mathbb{Z}_p \uparrow_{\mathcal{K}}^{\mathbb{G}_n} \cong \mathbb{Z}_p[[\mathbb{G}_n/k]]$

$$\text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p \uparrow_{\mathcal{K}}^{\mathbb{G}_n}, E_x) \xrightarrow{\text{cont}} \text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}[\mathbb{G}_n/k], E_x)$$

$$\simeq E_x e^{hk}$$

The case $n=1, p=2$:

$$1) \mathbb{G}_1 \simeq S_1 \cong \mathbb{Z}_p^*$$

2) $E_1 = 2\text{-adic } \mathcal{K}\text{-thy}$

$$3) E_1^{h\mathbb{G}_2} \simeq \hat{KO_i}$$

We have a fiber sequence

$$L_{K(1)} S^0 \rightarrow KU_2 \xrightarrow{+^3 - 1} KU_2$$

Mf

$$\text{SES} \quad 0 \rightarrow \mathbb{Z}_2[\mathbb{Z}_2] \xrightarrow{\iota} \mathbb{Z}_2[\mathbb{Z}_2] \rightarrow \mathbb{Z}_2 \rightarrow 0$$

There is iso

$$\mathbb{Z}_2[\mathbb{Z}_2] \simeq \mathbb{Z}_2[\mathbb{Z}_2]$$

$$+ \mapsto g - e$$

↑ top generator

We can rewrite this as

$$0 \rightarrow \mathbb{Z}_2 \xrightarrow{\Phi_1} \mathbb{Z}_2 \xrightarrow{\Phi_1} \mathbb{Z}_2 \rightarrow 0$$

Apply $\text{Hom}_{\mathbb{Z}_2}(-, E_*)$ and we get the

Sequence

$$0 \rightarrow E_* \rightarrow E_* E^{hG_2} \xrightarrow{hG_2} E_* E \rightarrow 0$$

$n=2, p=3$

Recall we had

$$0 \rightarrow G_n' \rightarrow G_n \xrightarrow{\text{det}} \mathbb{Z}_p^\times \rightarrow 0$$

~~$\mathbb{Z}/p-1$~~

For n coprime to p , we can write

$$G_n \cong G_n' \times \mathbb{Z}_p$$

① G_{24} is a subgroup of order 24

$$G_{24} = C_3 \times Q_8$$

Let $S = \frac{1}{2}(1 + \omega^5)$ $\omega = e^{\frac{2\pi i}{8}}$ order 3

Let ψ be gen of $\text{Gal}(\mathbb{F}_q/\mathbb{F}_3)$

$$G_{24} = \langle s, \omega^2, \omega\psi \rangle$$

$$\text{SD}_{16} = C_8 \rtimes \text{Gal}(\mathbb{F}_q/\mathbb{F}_3)$$

gen by ω & ψ

Q_8 Index 2 subgroup of SD_{16}

λ pull-back of sign-representation

Algebraic result

There is an exact sequence of $\mathbb{Z}_3[\mathbb{G}_2^!]$ -modules

$$0 \rightarrow \mathbb{Z}_3 \xrightarrow[G_{24}]{} \mathbb{Z}_3(\lambda) \xrightarrow[SD_{16}]{} \mathbb{Z}_3(\lambda) \xrightarrow[SD_{16}]{} \mathbb{Z}_3 \xrightarrow[G_{24}]{} 0$$

$\mathbb{Z}_3(\lambda)$ is \mathbb{Z}_3 but ω & Ψ act by mult by -1 .

Starting point is $H^*(S_2^1; \mathbb{F}_3)$ [Henn]

where S_2^1 is the 3-Sylow Subgroup of G_2

Why start here?

\mathbb{F}_3 -lined
dual

$$\text{Ext}_{\mathbb{Z}_3[[S_2^1]]}^q(M; \mathbb{F}_3) \cong \text{Tor}_{q-1}^{\mathbb{Z}_3[[S_2^1]]}(M; \mathbb{F}_3)^*$$

② $f: M \rightarrow N$ is a morphism of complete $\mathbb{Z}_3[[G]]$ mod

If $\widehat{f}: \widehat{F_3 \otimes f}: \widehat{F_3 \otimes M} \rightarrow \widehat{F_3 \otimes N}$ is surjective,
 $\mathbb{Z}[G]$ $\mathbb{Z}[G]$

then f is also surjective

$G = \text{f.g. profinite group}$.

We start w/ augmentation

$$0 \rightarrow N_1 \rightarrow \mathbb{Z}_3 \xrightarrow{\begin{pmatrix} G_1 \\ G_2 \\ G_{24} \end{pmatrix}} \mathbb{Z}_3 \rightarrow 0$$

We can calculate $\text{Ext}_{\mathbb{Z}_3[[S_2^1]]}(N, \mathbb{F}_3)$

using the long exact sequence in Ext

This allows you to define a map

$$0 \rightarrow N_2 \rightarrow I \xrightarrow{\begin{pmatrix} G_1 \\ SD_{16} \end{pmatrix}} \mathbb{Z}_3 \xrightarrow{f} N_1 \rightarrow 0$$

$f \otimes \mathbb{Z}_3$ is surjective
 $\mathbb{Z}_3[[S_2^1]]$

\Rightarrow we can calculate $\text{Ext}(N_3)$

$$0 \rightarrow N_3 \rightarrow \mathbb{Z} \xrightarrow{\begin{smallmatrix} G_2 \\ SD_{16} \end{smallmatrix}} \xrightarrow{f} N_2 \rightarrow 0$$

We want to show that

$$N_3 \cong \mathbb{Z}_3 \xrightarrow{\begin{smallmatrix} G_2 \\ G_{24} \end{smallmatrix}}$$

It turns out it suffices to show exists

$$f: N_3 \rightarrow \mathbb{Z}_3 \xrightarrow{\begin{smallmatrix} G_2 \\ G_{24} \end{smallmatrix}}$$

That is non-zero on Ext group.

It turns out you can construct such a

$$\text{map } \Rightarrow N_3 \cong \mathbb{Z}_3 \xrightarrow{\begin{smallmatrix} G_2 \\ G_{24} \end{smallmatrix}}$$

We can then "tensor" this up to G_2 itself.

$$\mathbb{Z}_3 \xrightarrow{\lambda} G_2 \curvearrowright E_* E^{h_{G_2}}$$

• λ is this pull-back

• Let e_λ be an idempotent in $\mathbb{Z}_3[SD_{16}]$

$$e_\lambda = \frac{1}{16} \sum_{g \in SD_{16}} \lambda(g) g^{-1}$$

We can form the telescope

$$E^\lambda := E \xrightarrow{e_\lambda} E \xrightarrow{e_\lambda} E \rightarrow \dots$$

$$E_* E^\lambda = \text{Hom}_{\mathbb{Z}_3[SD_{16}]}(\lambda, E_* E)$$

$$= \text{Hom}_{\mathbb{Z}_3[SD_{16}]}(\lambda, \text{Hom}^c((G_2, t_x))$$

$$\cong \text{Hom}_{\mathcal{D}_3}(\mathbb{G}_2) \left(\lambda \uparrow_{SD_{16}}^{G_2}, \text{Hom}(G_2, E) \right)$$

$$\cong \text{Hom}_{\mathcal{D}_3} \left(\lambda \uparrow_{SD_{16}}^{G_2}, E^* \right)$$

It turns out that $E^*E =$

$$E^* \left(\mathbb{C}^8 E^{hSD_{16}} \right)$$

We take the algebraic sequence and applying

$$\text{Hom}_{\mathcal{D}_3}(\mathbb{G}_2, E^*) \text{ we get}$$

$$(0 \rightarrow E_1 \rightarrow E, E^{hG_2}) \rightarrow E^*, \mathbb{C}^8 E^{hSD_{16}} \quad \text{④}$$

$$E^* E^{hG_2} \rightarrow E^*, \mathbb{C}^8 E^{hSD_{16}} \oplus E, E^{hG_2}$$

$$\rightarrow E_* E^{hG_{24}} \rightarrow 0$$

$$E_* \left(\sum^{\infty} E^{hG_{24}} \right) = E_* E^{hG_{24}}$$

We want to realise this topologically

Prop 27 (GML)

Let H_1, H_2 be closed subgroups of

G . If H_2 is finite then there is a commutative diagram

$n=2$

M_2

$$\begin{array}{ccc} \pi_* E_n ([G_n/M_1])^{hH_2} & \xrightarrow{\text{edge hom}} & (E_n)_{*} [G_n/M_1] \\ \downarrow \sim & & \downarrow \sim \\ \pi_* F(E_n^{hH_1}, E_n^{hH_2}) & \xrightarrow{\text{Morita}} & (E_*, E_*^{hH_1}, E_*^{hH_2}) \end{array}$$

E spectrum

$$X = \lim_i X_i$$

The $E(X)$ -holim $(E \wedge X_i)$

"lift" to this top map and show
that it is an isomorphism

The Toda bracket calculations work in a similar way.

Applications

- (1) Calculations of $\pi_*(L_{K(2)}V(0))$
- (2) Calculations of exotics in Φ^C
- (3) Chromatic splitting conjecture
- (4) B-C dual of $K(2)$ -local sphere
- (5) Rational homotopy of $K(2)$ -local sphere

* Agnes does work at $p=2$!

Mark Behrens

$$L_{K(2)} \text{tmf} \simeq E_2^{hG_{24}}$$

$$Q(\ell) = \text{Tot } (\text{TMF} \xrightarrow{\exists} \text{TMF} \times \text{TMF}_0(\ell) \\ \xrightarrow{\exists} \text{TMF}_0(\ell) \dots)$$

At $\ell=2$

there is a cofiber sequence

$$D_{K(2)} Q(2) \xrightarrow{D_F} L_{K(2)} \xrightarrow{S^0} Q(2)$$

Short res of $\Rightarrow H^*(G_2, E_2) \Rightarrow \pi_* L_{K(2)} S$
 $E_2 \wedge$

At ht $n=p-1$, Henn has another resolution