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Pic

Gross-Hopkins duality  
and Pic

- index  $\pi_*$  for  $*$   $\in \mathbb{Z}$

reason:  $\text{Pic}(S_p) = \mathbb{Z}$  generated by  $S^1$

$\text{Mf}$ :  $H_*$  by Künneth, map sphere in

- We still do this for  $K_n$

$$\bullet \mathbb{Z} \subset \text{Pic } K_n =: \text{Pic}_n = \left\{ \begin{array}{l} \text{equivalence classes of} \\ K_n = K\text{-local spectra under} \end{array} \right\} / \otimes$$

$\uparrow$

- When indexed on  $\text{Pic}_n$ ,  $\pi_*$  of a  $K$ -local  $S_p$  exhibit  
Gross-Hopkins duality

slides - Mazel-Gree

Outline

- $\text{Pic}_p$  for  $p > 2$
- Reduction to algebra for  $p \gg n$
- Brown-Comenetz duality and Gross-Hopkins duality and  
Cor
- sketch of proof

$p$ -complete

$$L_{K(1)} S \xrightarrow{1-\psi^\sigma} KU \rightarrow KU$$

$\sigma \in \mathbb{Z}_p^*$  gen  
 $\psi^\sigma$  Adams operation

$$E_1 \otimes L_{K(1)} S = E_1 S = E_1 \otimes \Rightarrow \mathbb{Z}_p^* \text{ acts trivially}$$

$\psi^\lambda$  acts on  $KU(S^{2n})$  by  $\lambda^n$

$$L_{K(1)} S^{2n} \rightarrow KU \xrightarrow{\sigma^n - \psi^\sigma} KU$$

$$\lambda \in \mathbb{Z}_p$$

$$\mathcal{S}(\lambda) \rightarrow KU_p^\wedge \xrightarrow{\psi^\sigma - \lambda} KU_p^\wedge$$

claim:  $\mathcal{S}(\lambda) \in \text{Pic}_1$

Pf:  $\mathcal{S}(\lambda) \hat{\wedge} \mathcal{S}(\lambda^{-1}) = \mathcal{S}$

# Prop (HMS)

TFAE 1.  $X \in \text{Pic}_n$

2.  $K(n)_{\ast} X$  is free  $\wedge$  over  $K(n)_{\ast}$

3.  $E_{n, \ast}^{\wedge} X$  is free  $\wedge$  over  $E_{n, \ast}$

$L_{K,n}$

$\psi^{\delta}$  acts by  $\lambda$  on  $E_{n, \ast}^{\wedge} S(k)$

$\Rightarrow$  They are all different

# Prop: (HMS)

by 3  $E_{n, \ast}^{\wedge} X$   
concentrated in even or odd degrees

$$0 \rightarrow \text{Pic}_1^0 \rightarrow \text{Pic}_1 \rightarrow \mathbb{Z}/2 \rightarrow 0$$

$S \downarrow \text{iso}$

$\mathbb{Z}_p^{\times}$

claim: Non-split

About:  $\mathbb{Z}_p^{\times} \cong \mathbb{Z}/(p-1) \oplus \mathbb{Z}_p$

So if non-split

$$\text{Pic}_1 = \mathbb{Z}/2(p-1) \oplus \mathbb{Z}_p$$

Let's construct the element of order  $2(p-1)$

$$p = \Sigma^{-1} \mathcal{S}(u) \quad u \in \mathbb{Z}_p^\times$$

$$\Sigma p = \mathcal{S}(u)$$

$$\mathcal{S}^{2(p-1)} = \mathcal{S}(u^{2(p-1)})$$
$$\mathcal{S}^{12}(\gamma^{(p-1)})$$

$$\Rightarrow u^{2(p-1)} = \gamma^{(p-1)}$$

$\Rightarrow u^2 = \gamma$  There are 2 by Hensel's Lemma

Because  $\gamma = \zeta_{(p+1)}$

↑ primitive  $(p+1)$ st root of unity

Can take  $u = \sqrt[p+1]{\phantom{x}}$

$$P = \Sigma^{-1} \left( \Sigma^{\sqrt[p+1]{\phantom{x}}} \right)$$

Prop: (HMS) Suppose  $2p-2 > \binom{\max}{n^2, 2n+2}$

Then

$$\left( \text{Pic}_n^0 \right)^{(E_n^\wedge)_0} \rightarrow \left( E_n - \sum_n \text{-modules} \right) / \text{iso}$$

↑  
non-extended

Morava  
stabilizer

is injective.

Pf: By sparsity for ANSS,  
let  $X \in \ker (E_n^\vee)_0$

$$\begin{array}{ccc}
 M \otimes E & \xrightarrow{g \otimes g} & M \otimes E \\
 \downarrow & & \downarrow \\
 M & \xrightarrow{g} & M
 \end{array}$$

$$H^{*,*}(S_n, E_{n*}(X)) \Rightarrow \mathbb{Z}_{p^n} \otimes_{\mathbb{Z}_p} \pi_* X$$

collapses by Sparcity result

$E_2$  term  $\cong E_2$  term of sphere

There's a class in  $E_2$  corr to

$$S \rightarrow L_{K(n)} S$$

(uses collapsing)

Corresponds to a map  $S \rightarrow X$

corresponds to  $\mathbb{Z} \rightarrow X$

gives  $L(\mathbb{Z}(n)) \mathbb{Z} \cong X$  □

Reduced determinant as  $S_n - E_n^*$  module

$S_n = \mathcal{O}_D^\lambda$  acts by lin maps on  $D$

$D$  vector space dim  $n^2$  over  $\mathbb{Q}_p$

$\det: S_n \rightarrow \mathbb{Z}_p^\times$  "unreduced det"

$\mathbb{Q}_{pn} \subset D$   $\det: S_n \rightarrow \mathbb{Q}_{pn}$

$\downarrow \cup$   
 $\mathbb{Z}_p^\times$

factors

through  $N: \mathbb{Z}_{pn}^\times \rightarrow \mathbb{Z}_p^\times$

Actually, let's use reduced det

$$\det: S_n \rightarrow \mathbb{Z}_p^{\times}$$

Brown-Cohen duality:

$E$  Spectrum

$IE$  Brown-Cohen dual

$$IE_n X := (E^n X)^V \leftarrow \begin{array}{l} \text{Pontryagin} \\ \text{dual} = \\ \text{Hom}(-, \mathbb{Q}/\mathbb{Z}) \\ = \text{Ext}^1(-, \mathbb{Z}) \end{array}$$

$$IE \simeq F(E, IS)$$

$\parallel$   
 $I$

"Brown-Cohen  
dualizing  
Spectrum"

Thm (Gross-Hopkins)  
 $L_{K(n)} I \in \text{Pic}_n$

In fact,  $L_{K(n)} I = \sum^{n^2-n} \mathbb{S}(\det)$

For  $p \gg h$

This has consequences for  $\Pi_*$  (Spectra)!

$$L_{K(n)} I \approx \text{For}(M_n, \mathbb{I})$$

$$\begin{aligned} \Pi_+(M_n X)^V &\cong [X, L_K I]_{-t} \cong \\ &[\mathbb{S}^{n^2-n-t}(\det)] \oplus DX \end{aligned}$$

Apply Gross-Hopkins  $= \Pi_{n-n^2-\det-t} DX$

• We need  $X$  to be a finite spectrum

If  $pX = 0$ , mod  $p$ ,  $\mathbb{S}(\det)$  becomes  
a "p-adic" sphere accessible w/  $V_n$ -periodicity

Thm (G-M) If  $F$  finite type  $n$  spectrum

which has a  $V_n$ -self map  $V$  with

$$K(n)_{\ast} \otimes V = V_n^{p^m}$$

$$pF = 0$$

$$\pi_+ (F)^V = \pi_{\alpha-+} (DF)$$

$$\alpha = \frac{2p^{nm}(p^n - 1)}{p-1} + n^2 - n$$

$X$  sm, proper alg var /  $\mathbb{R}$

$\mathcal{F}$  coh sheaf on  $X$

$K$  can bundle

$$H^i(X, \mathcal{F})^* \stackrel{K_n}{\cong} H^{n-i}(X, K \otimes \mathcal{F}^\vee)$$

Pontryagin  $\rightarrow$

$$K \longleftrightarrow I$$

$$\mathbb{D}E_n \cong \Sigma^{-n^2} E_n \quad \text{"looks like smooth manifold"}$$

by  $S_n$  PD group  
of this dimension

$$M_n E_n \cong \Sigma^{-n} E_n / I_n^\infty$$

$$\pi_* (E_n / I_n^\infty) \underset{\substack{\text{local} \\ \text{duality}}}{\cong} \int_{E_0 / \mathbb{Z}_p^n}^{\Sigma^{n-1}} \otimes (E_n)_{-t}$$

$$\begin{aligned} E_{+n} I &\cong \pi_+ (E \wedge L_K I) \\ &\cong \pi_+ (\Sigma^{n^2} \mathbb{D} E \wedge L_K I) \\ &\cong \pi_{+ - n^2} F(E, L_K I) \\ &\cong (\pi_{n^2 +} M E)^\vee \\ &\cong (\pi_{n^2 + n - t} E / I_n^\infty)^\vee \\ &\cong \int_{E_0}^{\Sigma^{n-1}} \otimes E_{+ - n^2 - n} \end{aligned}$$

$$\begin{aligned} \text{Need: } \Sigma^{n-1} &\cong E_{2n} [\det] \\ &\cong \omega^{\otimes n} [\det] \end{aligned}$$

GH do rigid geom to prove this