

4/23/13 - Peter Arndt

"The chromatic tower"

I. Chromatic convergence

L_n = loc w.r.t.

$E(n)$ or E_n or $K(0) \vee \dots \vee K(n)$

$$id \rightarrow \dots \rightarrow L_n \rightarrow L_{n-1} \rightarrow \dots \rightarrow L_0 = L_{\mathbb{Q}}$$

Height filtration on p-typical formal groups

Thm (Hopkins-Ravenel) X is finite p-local,

$$X \xrightarrow{\sim} \lim L_n X$$

• $\langle K(n) \rangle$ are minimal, so $\begin{array}{c} X \\ \downarrow L_E \\ L_n \rightarrow L_{n-1} \end{array}$

Reduction 1: Suffices to prove for $S(p)$

\hookrightarrow class satisfying C.C. is thick

$\hookrightarrow L_n$ are smashing i.e. for X a finite spectrum $L_n X = L_n S \otimes X$

Def: $C_n = \text{fib}(id \rightarrow L_n)$

$$\begin{array}{ccc} \varprojlim C_n X \rightarrow X \rightarrow \varprojlim L_n X & & \text{fiber sequence} \\ \downarrow & \parallel & \downarrow \\ C_n X \rightarrow X \rightarrow L_n X & & \end{array}$$

Reduction 2: Suffices to show

$$\left\{ \pi_m C_n \right\} \cong \{ 0^3 \}$$

Rmk: It suffices to show $\text{im}(\pi_m C_{n+s} \rightarrow \pi_m C_n)$ is trivial for large s

Input thms: $MU_* C_n S^{(p)} \rightarrow MU_* C_{n-1} S^{(p)}$ is zero

Pf: explicit computation

Rmk: This goes into the construction of the chromatic spectral sequence

Def $\widetilde{MU} = \text{fib}(S \rightarrow MU)$

$$\dots \rightarrow \widetilde{MU}^{\wedge 2} \rightarrow \widetilde{MU} \rightarrow S$$

The Adams - Novikov filtration of $\pi_* X$ is

$$F^S_{\pi_* X} = \text{im}(\pi_* \widetilde{MU}^{\wedge S} \wedge X \rightarrow \pi_* X)$$

$$\text{s.t. } \mu_{\ast} X \xrightarrow{\cong} \mu_{\ast} Y$$

Lemma: IF $f: X \rightarrow Y$ then $\pi_{\ast} f$ raises

A \wedge Filtration

Pf:

$$\begin{array}{ccc}
 S & \xrightarrow{\quad} & \\
 \downarrow & \searrow & \\
 X & \leftarrow \widetilde{\mu}^{\wedge s} \wedge X & \\
 \downarrow & & \\
 Y & \leftarrow \widetilde{\mu}^{\wedge s} \wedge Y & \leftarrow \widetilde{\mu}^{\wedge s+1} \wedge Y \\
 & & \downarrow
 \end{array}$$

claim: $\mu_{\ast} \widetilde{\mu}^{\wedge s} X \rightarrow \mu_{\ast} \widetilde{\mu}^{\wedge s+1} Y$
is 0 on π_{\ast}

Reason: $\mu_{\ast} \widetilde{\mu}$ is free

Pf: Have Künneth thm. Use hypothesis \square

$$So, \quad \pi_* C_{n+s} S_{(p)} \rightarrow \pi_* C_n S_{(p)}$$
$$F_S \pi_* C_n S_{(p)}$$

Reduction 3: A.N.F. on $\pi_m C_n S_{(p)}$ is finite

Dif: A map $X \rightarrow Y$ is called n-phantom

if \forall finite F $\dim F \leq n$

$[F, X] \rightarrow [F, Y]$ is 0

Dif: X is MU-convergent if $\forall n \exists s(n) \in S$

s.t. $\widehat{MU}^n s_n : X \rightarrow X$ is n-phantom

Reduction 4: X connected $\Rightarrow C_n X$ is MU-con⁺ A

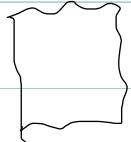
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3 facts: 1. X connected $\Rightarrow X$ MU-conv^t

2. MU-conv^t is thick

3. $\forall X, L_n X$ is MU-conv^t

$$\{L_n X\}_{n \geq 0} \xrightarrow{\sim} \{Tot^m E(n)^{p+1} X\}_{m \geq 0}$$



2. Layers

Chromatic fracture:

$$L_n X \longrightarrow k_n X$$



"Mayer-Vietoris"

$$L_{n-1} X \rightarrow L_{n-1}(k_n X)$$

Converse: $S =$ stable category

$L_n S = L_n$ local category

$$\begin{array}{ccc}
 L_n S \rightarrow L_{K(n)} S & & "L_n S = L_{n-1} S" \\
 \downarrow & \downarrow \text{loc at } L_{n-1} & \\
 \text{Arr } \underline{L_{n-1} S} \xrightarrow{\text{target}} L_{n-1} S & & L_{K(n)} S
 \end{array}$$

By chrom convergence, $S^{(p)} \xrightarrow{\text{fin}} \lim_{\leftarrow} L_n S^{(p)}$

Not essentially surjective

Let X_n be type n

$\dots \vee X_{n-1} \vee X_n \vee X_{n+1} \vee \dots$

Monochromatic
layers

M_n is Smashing

$$L_n L_{K(n)} X = L_{K(n)} X$$

$$\Rightarrow \begin{matrix} L_{K(n)} X \\ \downarrow \\ L_{n-1} L_{K(n)} X \end{matrix} \quad \text{is} \quad M_n L_{K(n)} X$$

Fiber square above

$$\Rightarrow M_n X \cong M_n L_{K(n)} X \quad (1)$$

$$\begin{matrix} (2) \quad L_{K(n)} M_n \rightarrow L_{K(n)} L_n \rightarrow L_{K(n)} L_{n-1} \\ \downarrow L \\ L_{K(n)} \end{matrix}$$

$$\Rightarrow L_{K(n)} M_n \cong L_{K(n)} \quad (2)$$

$$L_{K(n)} : M_n \mathcal{S} \xrightarrow[\sim]{} L_{K(n)} \mathcal{S} : M_n \quad \text{is a monoidal equivalence}$$

Warning: $T_X M_n \not\cong T_X L_{(C_n)}$

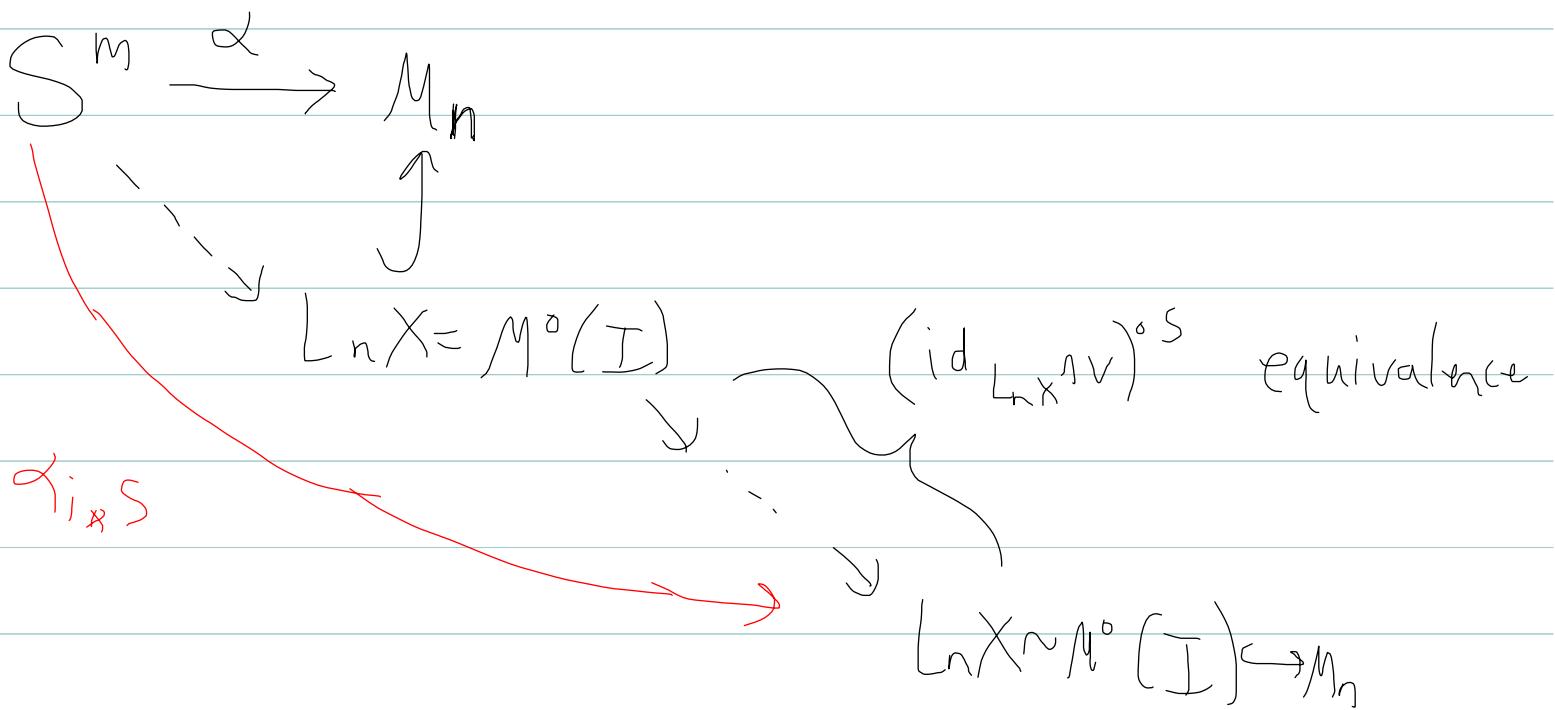
Periodic families in any M_n

$$M_n \times \simeq \underset{I}{\operatorname{colim}} \quad L_n \times \simeq M^o(I) \quad I = (i_0, \dots, i_{n-1})$$

finite

$$M^o(I) = M^o(p^{i_0}, \dots, p^{i_{n-1}})$$

admits a v_n -self map



get $\alpha_{i+s} \in T_{M_n, s + |v_n|} M_n \times \quad \alpha_0 = \alpha$

The α_n are asymptotically well defined

3. Spectral sequences

$$MU_* C_n S_{(p)} \xrightarrow{\quad} MU_* C_{n-1} S_{(p)}$$

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$$\text{Fact: } 0 \rightarrow BP_* S_{(p)} \hookrightarrow BP_* L_n S_{(p)} \rightarrow \rightarrow BP_* \sum C_n S_{(p)} \rightarrow 0$$

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$$0 \rightarrow BP_* S_{(p)} \rightarrow BP_* L_{n-1} S_{(p)} \rightarrow \rightarrow BP_* \sum C_{n-1} S_{(p)} \rightarrow 0$$

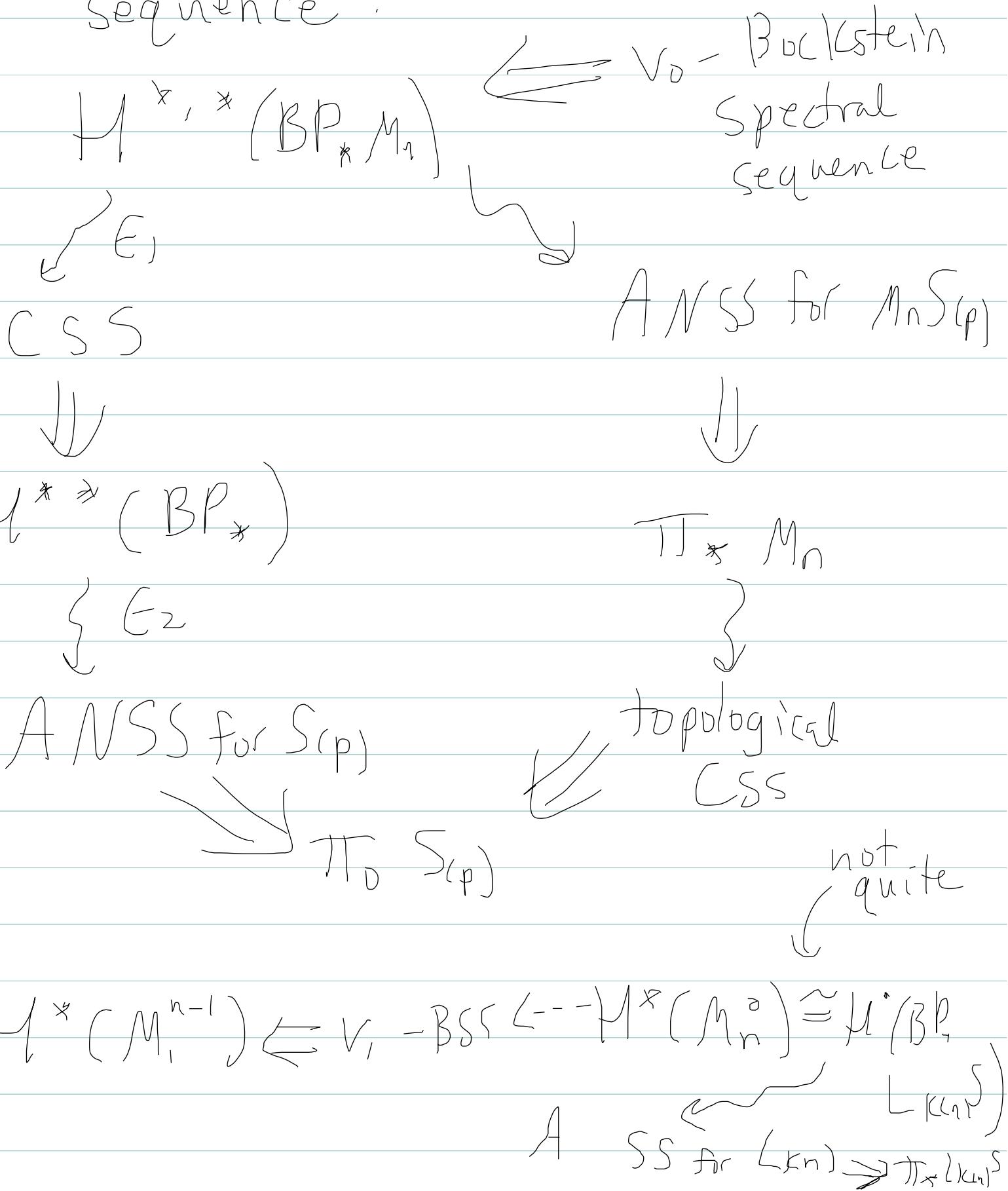
$$\begin{array}{ccc} & \sim & \\ BP_* M_n S_{(p)} & \longrightarrow & BP_* L_n S_{(p)} \\ \text{F.} & & \downarrow \text{zero} \\ (-1)^i & \rightsquigarrow & BP_* L_{n-1} S_{(p)} \end{array}$$

exact couple

$$\begin{array}{ccc} \oplus H^{i,j}(\widetilde{BP}_* L_n S_{(p)}) & \xrightarrow{\text{shift n degree}} & \oplus H^{i,j}(\widetilde{BP}_* L_n S_{(p)}) \\ i+j=n & \curvearrowleft & i+j=n \end{array}$$

$$\oplus H^{i,j}(\widetilde{BP}_* M_n S_{(p)})$$

This is chromatic spectral sequence.



$$\underset{\text{MCOR}}{\approx} \mathcal{M}_{\text{cts}}^*(G_n, E_n)_x \xrightarrow{\text{BSS}} \mathcal{M}_{\text{cts}}(G_i, E /_{I_i^{\infty}, r_i})$$



ANSS for $M_n S$

\approx MPPSS

$$\text{For } L_{(n)} \xrightarrow{\sim} f_n^{hb}$$



\Downarrow $\pi_* M_n S$