# Morava $E$-theory and change of rings

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## 1 Our goal

Some references: Devinatz's article on the change of rings theorem, Miller-Ravenel, Rezk's notes, Ravenel's green book.

Recall we're trying to compute $\pi_* S_{(p)}$ via $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$ and the ANSS. We can in turn compute the ANSS $E_2$-page via the CSS, which starts from $\mathrm{Ext}_{BP_*BP}(BP_*, v_n^{-1}BP_*/(p^\infty, \ldots, v_{n-1}^\infty))$; if we want, we can get at this using Bockstein spectral sequences which start from $\mathrm{Ext}_{BP_*BP}(BP_*, v_n^{-1}BP_*/I_n)$.

More generally, we want to compute $\mathrm{Ext}_{BP_*BP}(BP_*, M)$, where $M$ is a comodule satisfying

(i) $v_n^{-1}M = M$ (so $M$ lives on $\mathcal{M}_{FG}^{\leq n}$), or

(ii) $v_n^{-1}M = M$ and $I_n M = 0$ (so $M$ lives on $\mathcal{M}_{FG}^{=n}$).

One way to do this is to change Hopf algebroids to some better $(A, \Gamma)$ and compute $\mathrm{Ext}_\Gamma(A, A \otimes_{BP_*} M)$. For case (i), Morava $E$-theory is a good idea; for (ii), Morava $K$-theory is. In these cases, we can interpret our Ext group as the group cohomology of $\mathbb{S}_n$, the Morava stabilizer group.

(Kirsten: the parenthesized statements about $\mathcal{M}_{FG}$ after conditions (i) and (ii) above aren't equivalent to (i) and (ii), just implied by them.)

## 2 Defining $E_n$ and $\mathbb{S}_n$

### 2.1 Formal group laws

Recall the definitions of **formal group law**, **morphism** and **(strict) isomorphism**, **$p$-series**, and **$p$-typical FGL**. Let $k$ be a field of characteristic $p$ and $F$ an FGL over $k$.

**Lemma 1.** *If $f : F \to G$ is a nonzero morphism of FGLs, then $f(x) = g(x^{p^n})$ for some $n$ and some $g$ with $g(0) = 0$ and $g'(0) \neq 0$.*

In particular, $[p]_F(x) = g(x^{p^n})$ for some $g$ and some $n$, called the **height** of the FGL $F$. You can also read off the height from the kernel of the map from the $p$-typical Lazard ring $V$ that classifies $F$.

**Definition 2.** Let $k = \mathbb{F}_{p^n}$. Then $H_n$, the **Honda FGL of height** $n$, is the FGL classified by the map $V \to k$ sending $v_n$ to 1 and all other $v_i = 0$.

Equivalently, $H_n$ is defined by
$$[p]_{H_n}(x) = x^{p^n}.$$

### 2.2 The Morava stabilizer group

**Definition 3.** The **Morava stabilizer group** is the profinite group $\mathbb{S}_n = \mathrm{Aut}(H_n)$, where by automorphisms we mean strict self-isomorphisms.

We take a moment to establish some intuition about this group. It sits inside $\mathrm{End}(H_n)$, some of whose elements we can easily describe. For instance, there's a map $\mathbb{Z}_p \hookrightarrow \mathrm{End}(H_n)$ given by

$$\sum a_i p^i \mapsto \left[ x \mapsto \sum_{i=0}^{\infty}{}^{H_n} [a_i p^i]_{H_n}(x) \right]$$

There's a Frobenius endomorphism given by

$$S : x \mapsto x^p,$$

and if $\omega$ is a primitive element of $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$, there's an endomorphism

$$x \mapsto \omega x.$$

These endomorphisms satisfy some relations:

$$S^n = p; \qquad S\omega = \omega^\sigma S,$$

where $\sigma$ is the Frobenius automorphism on $\mathbb{F}_{p^n}$.

**Theorem 4** (Lubin-Tate)**.**

$$\mathrm{End}(H_n) = W(\mathbb{F}_{p^n})\langle S \rangle / (S^n = p, Sw = w^\sigma S \text{ for } w \in W(\mathbb{F}_{p^n})),$$

*where $W(\mathbb{F}_{p^n})$ is the Witt vectors and $\sigma$ is the lift of the Frobenius map to $W(\mathbb{F}_{p^n})$.*

Then, of course, we have $\mathbb{S}_n = \mathrm{End}(H_n)^\times$.

It's worth going into a bit more detail about the Witt vectors. These can be defined as the unique (up to isomorphism) complete local ring with residue field $\mathbb{F}_{p^n}$ such that if $(B, \mathfrak{m})$ is any complete local ring, then there exists a unique (continuous) map filling in the diagram

$$\begin{tikzcd} W(\mathbb{F}_{p^n}) \arrow[r, dashed, "\exists!"'] \arrow[d] & B \arrow[d] \\ \mathbb{F}_{p^n} \arrow[r] & B/\mathfrak{m}. \end{tikzcd}$$

Precisely, $W(\mathbb{F}_{p^n}) \cong \mathbb{Z}_p[x]/(q(x))$, where $q(x)$ is a lift of an irreducible factor of $x^{p^n-1} - 1 = 0$ to $\mathbb{Z}_p$.

### 2.3 Defining $E_n$

$E_n$ will have coefficient ring

$$(E_n)_* = W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]][u, u^{-1}]$$

where $|u_i| = 0$ and $|u| = -2$. The degree zero part is the universal deformation ring of $H_n$, in the following sense. Define the functor

$$\mathrm{Def}_{(\mathbb{F}_{p^n}, H_n)} : \text{ complete local rings } \to \text{ groupoids}$$

on a ring $B$ as the groupoid with objects

$$\mathrm{Def}_{(\mathbb{F}_{p^n}, H_n)}(B) = \{(G, i) : G \text{ a FGL on } B,\ i : \mathbb{F}_{p^n} \to B/\mathfrak{m}, \text{ such that } i_* H_n = \pi_* G\}.$$

Morphisms $(G_1, i_1) \to (G_2, i_2)$ only exist if $i_1 = i_2$, in which case they are strict isomorphisms $f : G_1 \to G_2$ such that $\pi_* f = 1_{B/\mathfrak{m}}$. (These are called **$*$-isomorphisms**.)

**Theorem 5** (Lubin-Tate)**.** $\mathrm{Def}_{(\mathbb{F}_{p^n},H_n)}(B)$ *splits as a disjoint union of groupoids* $\mathrm{Def}_{(\mathbb{F}_{p^n},H_n)}(B)_i$ *(the set of pairs* $(G,i)$ *with fixed* $i$*), with*

$$\pi_0(\mathrm{Def}_{(\mathbb{F}_{p^n},H_n)}(B)_i) = \mathfrak{m}^{\times(n-1)}$$

*and*

$$\pi_1(\mathrm{Def}_{(\mathbb{F}_{p^n},H_n)}(B)_i) = *.$$

*Thus,* $\pi_0(\mathrm{Def}_{(\mathbb{F}_{p^n},H_n)}(\cdot))$ *is corepresented by* $W(\mathbb{F}_{p^n})[[u_1,\ldots,u_{n-1}]]$.

**Corollary 6.**

1. $\mathbb{S}_n$ *acts on* $W(\mathbb{F}_{p^n})[[u_1,\ldots,u_{n-1}]]$*, and*

2. $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ *acts on this ring as well, by changing the map* $i:\mathbb{F}_{p^n}\to B/\mathfrak{m}$.

*These combine to give an action of the extended Morava stabilizer group* $\mathbb{G}_n = \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$.

Now, $E_n$ is a $BP_*$-module, where $v_i$ acts by $u_i u^{1-p^i}$ for $i \le n-1$, $u^{1-p^n}$ for $i=n$, and 0 otherwise. Thus the LEFT applies, and we get a spectrum $E_n$, called **Morava $E$-theory**.

## 3 Change of rings theorems

Recall that we want to compute $\mathrm{Ext}_{BP_*BP}(BP_*,M)$ in cases (i) and (ii) above. First suppose we're in case (ii), where $v_n^{-1}M = M$ and $I_nM=0$. Miller-Ravenel showed that in this case,

$$\mathrm{Ext}_{BP_*BP}(BP_*,M) \cong \mathrm{Ext}_{\Sigma(n)}(K(n), K(n)\otimes_{BP_*} M).$$

Here $K(n)$ is Morava $K$-theory and $\Sigma(n) = K(n)\otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)$. Likewise, in case (i), where we just have $v_n^{-1}M = M$,

$$\mathrm{Ext}_{BP_*BP}(BP_*,M) \cong \mathrm{Ext}_{\widehat{U}(n)}(\widehat{E}(n), \widehat{E}(n)\otimes_{BP_*} M).$$

Here $\widehat{E}(n)$ is completed periodic Johnson-Wilson theory, with coefficient ring $\mathbb{Z}_p[[u_1,\ldots,u_{n-1}]][u,u^{-1}]$; $\widehat{U}(n)$ is defined similarly to $\Sigma(n)$, though it's completed now. We can write these both in terms of group cohomology. In case (i), we have

$$\mathrm{Ext}_{BP_*BP}(BP_*,M) \cong H_c^*(\mathbb{S}_n, E_n\otimes_{BP_*} M)^{\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)};$$

in case (ii), we have a map

$$\mathrm{Ext}_{BP_*BP}(BP_*,M) \to H_c^*(\mathbb{S}_n, \mathbb{F}_p\otimes_{BP_*} M)^{\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)},$$

which becomes an isomorphism after doing some things with the grading.

All this follows from two types of general change-of-rings theorems. Let $f:(A,\Gamma)\to(B,\Sigma)$ be a map of Hopf algebroids. We then have a pair of functors

$$\Gamma\text{–}\mathsf{Comod} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \Sigma\text{–}\mathsf{Comod}$$

given by $f^*(M) = B\otimes_A M$ and $f_*(N) = (\Gamma\otimes_A B)\square_\Sigma N$. If we'd like, we can view these as pullback and pushforward of quasicoherent sheaves on the stacks $\mathcal{M}_{(A,\Gamma)}$ and $\mathcal{M}_{(B,\Sigma)}$.

The first change-of-rings theorem is Miller-Ravenel's 'push-pull' theorem, which says that in the pair of maps

$$\mathrm{Ext}_\Gamma(A,M) \to \mathrm{Ext}_\Gamma(A, f_*f^*M) \to \mathrm{Ext}_\Sigma(B, f^*M),$$

the first is an isomorphism under conditions on $M$, and the second is an isomorphism under conditions on $f$.

The second comes from the concept of equivalence of Hopf algebroids. A Hopf algebroid can be viewed as a functor from rings to groupoids, and a map $f$ of Hopf algebroids induces a natural transformation of functors.

**Definition 7.** $f$ is an **equivalence of Hopf algebroids** if the associated natural transformation admits an inverse, up to natural equivalence. Equivalently, $f$ is an equivalence if it induces an equivalence on the associated stacks.

In this case, we have a change-of-rings isomorphism $\mathrm{Ext}_\Gamma(A, M) \cong \mathrm{Ext}_\Sigma(B, f^*M)$.