Periodicity

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April 22, 2013

Recall that the CSS arises from a filtration

$$BP_* \to p^{-1}BP_* \to v_1^{-1}BP_*/p^\infty \to \cdots$$

that comes from algebraic periodicity in BP_*. One is led to wonder whether this algebraic periodicity is realized topologically.

Definition 1. Let X be finite and p-local and $v : \Sigma^k X \to X$ a self map. v is a v_n**-self map** if $K(m)_* v$ is nilpotent for $m \neq n$, an isomorphism for $m = n \neq 0$, and multiplication by a nonzero rational number for $m = n = 0$.

Theorem 2 (Periodicity theorem, Devinatz-Hopkins-Smith)**.** *X admits a v_n-self map iff it is $K(n-1)_*$-acyclic.*

The proof of this entirely relies on the nilpotence theorem in its Morava K-theory incarnation, which we just proved.

Let $\mathcal{C}_0$ be the category of p-local finite spectra and $\mathcal{C}_n$ the full subcategory of $K(n-1)_*$-acyclic spectra in $\mathcal{C}_0$.

Theorem 3 (Ravenel, Mitchell-Smith)**.** *There is a chain of proper inclusions*

$$\mathcal{C}_\infty \subsetneq \cdots \subsetneq \mathcal{C}_n \subsetneq \cdots \subsetneq \mathcal{C}_1 \subsetneq \mathcal{C}_0.$$

Definition 4. A full subcategory $\mathcal{C}$ of $\mathcal{C}_0$ is **thick** if it is closed under weak equivalences, cofiber sequences, and retracts.

Theorem 5 (Thick subcategory theorem, Hopkins-Smith)**.** *If $\mathcal{C} \subseteq \mathcal{C}_0$ is thick, then it is equal to some $\mathcal{C}_n$.*

Proof. This follows from the nilpotence theorem, and is in fact equivalent to it. $\square$

Definition 6. A property of finite spectra is **generic** if the category of spectra having that property is thick.

Let $\mathcal{V}_n \subseteq \mathcal{C}_0$ be the full subcategory of spectra admitting v_n-self maps. We want to show that $\mathcal{V}_n = \mathcal{C}_n$. First, $\mathcal{C}_{n+1} \subseteq \mathcal{V}_n$: any self map of a $K(n)_*$-acyclic spectrum is a v_n-self map. Second, $\mathcal{V}_n \subseteq \mathcal{C}_n$. Suppose X admits a v_n-self map but $K(i)_* X \neq 0$ for some $i < n$. If Y is the cofiber of the v_n-self map, then $K(n)_* Y = 0$, but $K(i)_* Y \neq 0$, contradicting Ravenel's theorem above.

Third, we must show that the property of admitting a v_n-self map is generic, which we will prove below. This will prove that $\mathcal{V}_n = \mathcal{C}_{n+1}$ or $\mathcal{C}_n$. The fourth step, which completes the proof, is to exhibit an $X \in \mathcal{V}_n \cap \mathcal{C}_n \setminus \mathcal{C}_{n+1}$. This is technical and will not be discussed today.

Lemma 7. *Let $f : \Sigma^k X \to X$ be a v_n-self map. Then there exist i, j with $K(m)_* f^i$ equal to multiplication by v_n^j for $m = n$ and 0 for $m \neq n$.*

Lemma 8. *Under the above conditions, there exists i such that $f^i \in Z(\mathrm{End}(X))$.*

Lemma 9. *Let g be another v_n-self map. Then there exist i, j with $f^i = g^j$.*

Lemma 10. *Let f be as above, g a v_n-self map of Y. Then there exist i, j such that for all $h : X \to Y$, the square*

$$\begin{tikzcd} \Sigma^M X \arrow[r, "\Sigma^M h"] \arrow[d, "f^i"'] & \Sigma^M Y \arrow[d, "g^j"] \\ X \arrow[r, "h"] & Y \end{tikzcd}$$

commutes.

Now, since X is finite, it has a Spanier-Whitehead dual DX with $[X, Y]_* \cong [S, DX \wedge Y]_*$ for all Y. In particular, $\mathrm{Hom}_{K(m)_*}(K(m)_*X, K(m)_*Y) \cong K(m)_*(Y \wedge DX)$. Our strategy will be to dualize and apply the nilpotence theorem a lot.

Definition 11. Let R be a finite ring spectrum of the form $(X \wedge DX)$ and $\alpha in \pi_* R$. Then α is a v_n**-element** if $K(m)_*\alpha$ is multiplication by a unit for $m = n$ and nilpotent for $m \neq n$.

There's a clear correspondence between v_n-self maps of X and v_n-elements of $\pi_*(X \wedge DX)$.

Proof of Lemma 7. We can restate the lemma as saying that if $\alpha \in \pi_* R$ is a v_n-element, then there exist i, j such that $K(m)_*\alpha^i = v_n^j$ for $m = n$ and 0 for $m \neq n$. For large m, $K(m)_*X \cong H\mathbb{F}_{p*}X \otimes_{\mathbb{F}_p} K(m)_*$, and a map $f : X \to Y$ has $K(m)_*f = H\mathbb{F}_{p*}f \otimes 1_{K(m)_*}$. Thus if $K(m)_*\alpha$ is nilpotent for $m \gg 0$, then $H\mathbb{F}_{p*}\alpha$ is nilpotent for $m \gg 0$, so $H\mathbb{F}_{p*}\alpha = 0$ for $m \gg 0$. In particular, $K(m)_*\alpha = 0$ for all but finitely many m, and since α is a v_n-element, raising it to a sufficiently high power gives $K(m)_*\alpha = 0$ for all $m \neq n$. Finally, $K(n)_*R/(v_n - 1)$ has a finite group of units, and α is a unit in this ring; thus, raising it to some power, we get $\alpha = 1$ in this ring and thus $\alpha = v_n^j$, as desired. □

Proof of Lemma 8. The dual statement of this lemma is that for α a v_n-element, there exists i such that $\alpha^i \in Z(\pi_* R)$. We first establish an auxiliary lemma:

Lemma 12. *Let x, y be commuting elements of a $\mathbb{Z}_{(p)}$-algebra such that $(x - y)$ is nilpotent and torsion. Then for $N \gg 0$, $x^{p^N} = y^{p^N}$.*

This follows from the binomial theorem.

Now let $\ell(\alpha)$ and $r(\alpha)$ be multiplication on the left and right by α, respectively. Then $\ell(\alpha) - r(\alpha)$ has finite order, and since α is central in K-theory, $K(m)_*(\ell(\alpha) - r(\alpha)) = 0$ for all m. Thus by the nilpotence theorem, $\ell(\alpha) - r(\alpha)$ is nilpotent, and so $\ell(\alpha)^{p^N} = r(\alpha)^{p^N}$, proving that some power of α is in $Z(\pi_* R)$. □

Proof of Lemma 9. The dual statement is that if $\alpha, \beta \in \pi_* R$ are v_n-elements, then after taking high enough powers, they are equal.

This is the same proof as the previous one: after taking high enough powers, $K(m)_*(\alpha - \beta) = 0$ for all m, and α, β are central. So $\alpha - \beta$ is nilpotent and finite order, and after taking more powers, it is zero. □

Proof of Lemma 10. Let v_X and v_Y be v_n-self maps on X and Y respectively, and $h : X \to Y$. We have a square

$$\begin{tikzcd} \Sigma^M X \arrow[r, "\Sigma^M h"] \arrow[d, "v_X"'] & \Sigma^M Y \arrow[d, "v_Y"] \\ X \arrow[r, "h"] & Y \end{tikzcd}$$

and dualizing gives a diagram

$$\begin{tikzcd} S^M \arrow[r] & Y \wedge DX \arrow[d, shift right, "v_Y \wedge 1_{DX}"'] \arrow[d, shift left, "1_Y \wedge Dv_X"] \\ & Y \wedge DX. \end{tikzcd}$$

One can show that the parallel maps are again v_n-self maps, so taking high enough powers gives the desired result. □

Proof that $\mathcal{V}_n$ is thick. First we show that it's closed under cofiber sequences. Clearly, X admits a v_n-self map iff ΣX does. Thus it suffices to show that if $X \to Y \to Z$ is a cofiber sequence with X and Y admitting v_n-self maps, then Z does as well. By lemma 4, there's a diagram

$$\begin{tikzcd}
\Sigma^M X \arrow[r] \arrow[d, "v_X"'] & \Sigma^M Y \arrow[r] \arrow[d, "v_Y"'] & \Sigma^M Z \arrow[d, dashed, "v_Z"'] \\
X \arrow[r] & Y \arrow[r] & Z
\end{tikzcd}$$

and if we let v_Z be any map filling in the diagram, it's easy to see that v_Z is a v_n-self map as well.

Second, we show that $\mathcal{V}_n$ is closed under retracts. Let $Y \xrightarrow{i} X \xrightarrow{r} Y$ be a retraction, and let v_X be a v_n-self map of X commuting with ir. Then one can check that $rv_X i$ is a v_n-self map of Y. □

As we said, the last step is to show that $\mathcal{V}_n$ is not equal to $\mathcal{C}_{n+1}$, which is a difficult argument, using vanishing lines and stuff.

One nice corollary is

Theorem 13. *Let $X \in \mathcal{C}_n$. The map $Z([X,X]_*) \to \mathbb{F}_p[v_n]$ has kernel consisting of nilpotent elements, and image containing some v_n^j.*