Introduction

Mark Behrens

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1 Motivation

Mark is going to get the juices flowing as it were.

[picture of $\pi_* S$ at the prime 2 taken from Hatcher's website - computation by Mahowald-Tangora-Kochman]

Each dot represents a $\mathbb{Z}/2$, and vertical lines indicate nontrivial additive extensions. Diagonal and horizontal lines represent multiplication by η and ν. There is no vertical axis.

There's a sense of pattern to this diagram, but it's kind of a mess, and was a mess until Jack Morava started studying chromatic homotopy theory to organize it.

[picture of $\pi_* S$ at the prime 3]

[picture of $\pi_* S$ at the prime 5]

These get easier to do as p increases, and more regular, as can be seen from these pictures alone.

Let's talk about some of these patterns. The first pattern you see is $\operatorname{Im} J$, where J is a homomorphism from $\pi_* SO$ to $\pi_* S$. At prime 2, this captures the 8-periodic pattern on the bottom rows, though it misses some dots. Likewise, there's a periodic pattern coming from $\operatorname{Im} J$ at every prime, which is probably the easiest part of $\pi_* S$ to understand.

Definition 1. $J : \pi_* SO \to \pi_* S$ is induced by the colimit of the maps $SO(n) \to \Omega^n S^n$, each of which is defined as follows. Given $A \in SO(n)$, A can be viewed as a map $\mathbb{R}^n \to \mathbb{R}^n$, and taking one-point compactifications gives an element of $\Omega^n S^n$.

Now, Bott periodicity tells us that $\pi_* SO$ is $\mathbb{Z}/2$ in dimensions 0 and 1 mod 8, $\mathbb{Z}$ in dimensions 3 and 7 mod 8, and 0 in other dimensions. Thus, $\operatorname{Im} J$ has some sort of 8-periodicity to it.

(In fact, the J-homomorphism can be realized as a map of spectra $\Sigma^{-1} bso \to S$, where bso is the connective cover of real K-theory with π_0 and π_1 killed. Unfortunately, this point of view isn't terribly useful, since taking connective covers and desuspending has destroyed all ringness in the source.)

Theorem 2 (Adams)**.** *In dimension $4k-1$, $\operatorname{Im} J$ is a group of order the denominator of $\frac{B_k}{4k}$, where B_k is the kth Bernoulli number.*

So this is understandable, but also number-theoretic and complicated globally. Things will be easier for us if we localize at a prime p, which we do from now on.

There's also a spectrum-level version of this theory localized at p. Namely, if $KO^{\wedge}_p$ is the p-completion of KO, there are p-local Adams operations $\psi^\ell : KO_p^{\wedge} \to KO_p^{\wedge}$, and the fiber of $\psi^\ell - 1$ is defined to be J_p, where ℓ is any prime different from p, i.e. a topological generator of $\mathbb{Z}_p^\times$. We get a diagram

$$\begin{tikzcd}
 & \Sigma^{-1} bso \arrow[dl] \arrow[dr] & & \\
\Sigma^{-1} KO_p^{\wedge} \arrow[d] & & S \arrow[d] \arrow[dll, dashed] & \\
J_p \arrow[rr] & & KO_p^{\wedge} \arrow[r, "\psi^\ell - 1"'] & KO_p^{\wedge}.
\end{tikzcd}$$

Here the top right map is the J-homomorphism, and the right map is the Hurewicz homomorphism, which is fixed by ψ^ℓ, so that it lifts to J_p; the diagram shows that the J-homomorphism is just the fiber of this map.

Theorem 3 (Adams-Baird)**.** $S_{K/p} \simeq J_p$.

2 Primes of homotopy theory

In number theory and algebra, one often studies problems by localizing them at each prime p, as well as rationalizing them (localizing at the prime 0). This corresponds to a chain of inclusions $\mathrm{Spec}(\mathbb{Q}) \hookrightarrow \mathrm{Spec}(\mathbb{Z}_{(p)}) \hookrightarrow \mathrm{Spec}(\mathbb{Z})$, but we can go no further, corresponding to the fact that $\mathbb{Z}$ has Krull dimension 1.

On the other hand, in topology, the sphere has infinite 'Krull dimension'. Thus there are localizations $S \to S_{(p)} \to S_{\mathbb{Q}}$, but also infinitely many intermediate localizations.

Let's introduce some cohomology theories that will haunt us this week (or for the rest of our lives). Recall that BP is a spectrum with

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots].$$

Johnson-Wilson theory is given by

$$E(n) = BP/(v_{n+1}, v_{n+2}, \ldots)[v_n^{-1}],$$

and Morava K-theory is given by

$$K(n) = E(n)/(p, v_1, v_2, \ldots).$$

By convention, $K(0) = E(0) = H\mathbb{Q}$. We get an infinite tower of localizations

$$S \to S_{(p)} \to \cdots \to S_{E(2)} \to S_{E(1)} \to S_{\mathbb{Q}},$$

called the **chromatic tower**. (It's a consequence of the nilpotence theorem that these are the 'only primes,' but making this statement rigorous is a little difficult. Hopkins and Devinatz have a notion of 'field spectra' which can be used for this purpose. Another way of thinking about this is via the thick subcategory theorem – if we think of primes as things with respect to which a spectrum can be completed, then this theorem implies that the only such completions are those with respect to Morava K-theories.)

(Toby: the Bousfield classes of Morava K-theories are minimal, which is another argument for the fact that they're similar to maximal primes.)

We can now filter $\pi_* S_{(p)}$ by letting the nth layer of the filtration be $\ker(\pi_* S_{(p)} \to \pi_* S_{E(n-1)})$. This is called the **$v_n$-periodic layer**. The v_1-periodic layer is just $\operatorname{im} J$, which has fundamental period $2(p-1)$ for odd p. Likewise we can pick out the v_2- and v_3-periodic layers, with periods $2(p^2-1)$ and $2(p^3-1)$, and so on.

(Arnav: can there be nontrivial additive extensions between different layers? Mark: I'm not sure.)

3 Periodicity in the layers

Definition 4. The **nth monochromatic layer** $M_n S$ is the fiber of $S_{E(n)} \to S_{E(n-1)}$.

Theorem 5 (Nilpotence theorem, Hopkins-Devinatz-Smith)**.** *Let $I = (i_0, \ldots, i_{n-1})$ be a sequence of integers. Then for a cofinal set of $I \in \mathbb{N}^n$, a finite complex M_i exists with $BP_* M_I = BP_*/(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$, and for i_n sufficiently large, there is a self-map*

$$v_n^{i_n} : \Sigma^{2 i_n (p^n - 1)} M_I \to M_I$$

which is an E_n-isomorphism, and thus non-nilpotent.

Let M_I^0 be a desuspension of M_I so that its top cell is in dimension zero. Then it's in fact true that $M_n S = \varinjlim_I (M_I^0)_{E(n)}$.

We get a diagram

$$\begin{tikzcd} S_{(p)} \arrow[d] & & \\ S_{E(n)} \arrow[d] & M_nS = \varinjlim (M_I^0)_{E(n)} \arrow[l] & (M_I^0)_{E(n)} \arrow[l] \\ S_{E(n-1)} & & \end{tikzcd}$$

Thus if $x \in \pi_* S_{(p)}$, we can lift its image to $\pi_* M_n S$, and this comes from $\pi_*(M_I^0)_{E(n)}$ for some I, which is known to be periodic. This allows us to construct infinite families in the stable homotopy groups of spheres. (Note that the nilpotence theorem gives us periodicity in global M_I^0, but it's often very difficult to find explicit self-maps giving us these infinite families. If we had a map down to the sphere that would detect the beta family, ... whoaaaaaaaaa. I'm beginning to salivate.)

4 Completions

Let M be a finitely generated abelian group. There's a pullback square

$$\begin{tikzcd} M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_p M_p\hat{} \arrow[d] \\ M_{\mathbb{Q}} \arrow[r] & \left(\prod_p M_p\hat{}\right)_{\mathbb{Q}} . \end{tikzcd}$$

A similar **arithmetic square** exists in homotopy theory:

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_p X_p\hat{} \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & \left(\prod_p X_p\hat{}\right)_{\mathbb{Q}} . \end{tikzcd}$$

Likewise, there's a **chromatic fracture square** arising from the chromatic tower:

$$\begin{tikzcd} X_{E(n)} \arrow[r] \arrow[d] & X_{K(n)} \arrow[d] \\ X_{E(n-1)} \arrow[r] & (X_{K(n)})_{E(n-1)} \end{tikzcd}$$

5 Moduli interpretation

The moduli interpretation comes from the Adams-Novikov spectral sequence, which is a spectral sequence

$$\mathrm{Ext}_{MU_*MU}(MU_*, MU_*) \Rightarrow \pi_* S.$$

The Quillen-Lazard theorem tells us that this Ext term can be reinterpreted as $H^*(\mathcal{M}_{FG})$, where $\mathcal{M}_{FG}$ is the moduli stack of one-dimensional formal groups over $\mathrm{Spec}(\mathbb{Z})$. The chromatic tower then reappears as a description of this stack, as follows. First, we base change to characteristic p; formal groups over $\mathbb{F}_p$ have a height, so we can filter $\mathcal{M}_{FG} \otimes \mathbb{F}_p$ via the closed substacks $\mathcal{M}_{FG}^{\geq n}$, which are the moduli stacks of formal group laws of height at least n. We then let $\mathcal{M}_{FG}^{\leq n} = (\mathcal{M}_{FG})_{(p)} - \mathcal{M}_{FG}^{\geq n+1}$, which is an open subscheme. The chromatic tower now becomes

$$\mathcal{M}_{FG} \hookleftarrow (\mathcal{M}_{FG})_{(p)} \hookleftarrow \cdots \hookleftarrow \mathcal{M}_{FG}^{\leq 2} \hookleftarrow \mathcal{M}_{FG}^{\leq 1} \hookleftarrow (\mathcal{M}_{FG})_{\mathbb{Q}}.$$

We thus get a spectral sequence $H^*(\mathcal{M}_{FG}^{\leq n}) \Rightarrow \pi_* S_{E(n)}$.

6 Formal moduli

It's worth pointing out that $M_n(S_{K(n)}) \simeq M_n S$, so to understand the sphere monochromatically, it suffices to understand its $K(n)$-localizations. Let $\mathcal{M}_{FG}^{=n}$ be the closed subscheme $\mathcal{M}_{FG}^{\leq n} \cap \mathcal{M}_{FG}^{\geq n}$ of $\mathcal{M}_{FG}^{\leq n}$. There's a spectral sequence $H^*((\mathcal{M}_{FG}^{\leq n})^\frown_{\mathcal{M}_{FG}^{=n}}) \Rightarrow \pi_* S_{K(n)}$, where this completion is a formal neighborhood of $\mathcal{M}_{FG}^{=n}$ in $\mathcal{M}_{FG}^{\leq n}$. Thus, to understand the $K(n)$-local sphere, we should try to understand deformations of formal groups, which was done by Lubin-Tate.

As a stack, $\mathcal{M}_{FG}^{=n} \otimes \overline{\mathbb{F}_p}$ is a single point, corresponding to the **Honda formal group** H_n. Its automorphisms are $\mathbb{S}_n$, the n**th Morava stabilizer group**. Lubin-Tate showed that the deformations of H_n are classified by $(E_n)_0 = W(\overline{\mathbb{F}_p})[[u_1, \ldots, u_{n-1}]]$. There's also a spectrum E_n wih $\pi_* E_n = (E_n)_0[u^{\pm 1}]$.

By the Morava change of rings theorem,

$$H^*((\mathcal{M}_{FG}^{\leq n})^\frown_{\mathcal{M}_{FG}^{=n}}) \cong H_c^*(\mathbb{G}_n; \pi_* E_n),$$

where the right-hand side is continuous cohomology of the profinite group $\mathbb{G}_n$, where $\mathbb{G}_n = \mathbb{S}_n \rtimes \mathrm{Gal}(\overline{F_p}/\mathbb{F}_p)$, the **extended Morava stabilizer group**. (Defining this uses the fact that H_n can be defined over $\mathbb{F}_p$.)

In fact, this instance of the Morava change of rings theorem can be realized topologically as the statement that $\mathbb{G}_n$ acts on E_n, with homotopy fixed points $E_n^{\mathrm{h}\mathbb{G}_n} \simeq S_{K(n)}$. Then the spectral sequence $H_c^*(\mathbb{G}_n; \pi_* E_n) \Rightarrow \pi_* S_{K(n)}$ is just a homotopy fixed point spectral sequence.

Example 6. Let $n = 1$. Then $E_1^{\mathrm{h\,Gal}} \simeq KU_p^\frown$ is acted on by $\mathbb{S}_1 = \mathbb{Z}_p^\times$, and this action is just given by the Adams operation ψ^ℓ. Thus, we recover Adams's theory of the image of J.

7 Bad primes

The cohomology groups $H_c^*(\mathbb{G}_n; \pi_* E_n)$ can, *in principle*, be computed. This is good news. But there's bad news as well. For every chromatic level n, there's a finite set of **bad primes**.

Definition 7. The **chromatic conductor** of the prime p at chromatic level n is the largest r such that $\mathbb{S}_n$ has an element of order p^r.

We write $n = (p-1)p^{r-1}s$ for some s prime to p. If n is divisible by $(p-1)$, it's bad, and if it's divisible by $p(p-1)$, it's even worse – larger chromatic conductors imply more badness.

Badness results in several problems, such as irregular periods and extra exotic p-torsion. For example, when $n = 1$, the fundamental period should be $2(p-1)$, but at the bad prime $p = 2$, the period is 8 instead. When $n = 2$, the fundamental period should be $2(p^2-1)$, but at the bad primes 2 and 3, it's instead 192 and 144 respectively.

One way to deal with this is to build spectra that detect the badness. For example, if $H \leq \mathbb{G}_n$ is a subgroup that contains $\mathbb{Z}/p^r$, we can form the homotopy fixed points E_n^{hH}, which should contain all the badness of the prime p.

The most basic example is $KO \subseteq KU$, which contains all the 8-periodicity of 2-local K-theory! Other examples are TMF and $EO(n)$.

[more pictures are shown]