Unity of Topological Field Theories

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About 15 years ago Witten introduced three classes of quantum field theories in

dimensions.

2, 3, 4

(topological strings)

Chern-Simons gauge theory (knot invariants)

N=2 SYM (twisted)

Donaldson invariants

\[ \Sigma \to M \]

Now we know they are all deeply connected via strings.
Topological Strings

Open

Closed

large N dualities

$\Sigma \neq 0$

$\phi = 0$

A

Target Worldsheet

Chern Simons theory + knots

Gromov-Witten

$\Sigma \subset \text{Lagrangian}$

Quantum Cohomology

Gromov-Witten theory; $\Sigma = 0$

B

Mirror Symmetry

Holomorphic C.S. (d)

BF theory @ 2d

Matrix model

Feynman graphs

Kodaira-Spencer theory + quantum V.H.S.

Feynman graphs of K.S. theory

Superstring on $CY^3 \times \mathbb{R}^4$

$N = 1, d = 4$

$N = 2, d = 4$

More realistic physics

$4 + 6 = 10 !$

Donaldson invariants
\( S_{cs} = \frac{1}{k} \text{SCSA}(\mathcal{M}) \)

\( \mathcal{M}^3 \)

\( \text{target} \)

\( \mathcal{T}^* \mathcal{M}^3 \)

\( U(N) \)

Chern-Simons on \( \mathcal{M}^3 \)

(multiplicity \( N \) on \( \mathcal{M}^3 \))

\( N^h g^2 - 2 + h \)

\( h \): holes

\( g \): genus

\( \Delta f = 0 \)

\( \mathcal{L}_{3} \)

Lagrangian

Worldsheet
$$\Theta \Sigma \in \left[ S^3 \bigcup L \right] \frac{N}{M}$$

\rightarrow\text{ knot invariants on } S^3.\n
U(N), \; g_s = \frac{2n^2}{k+N}; \text{ M\to Rep. on } L.
Interesting class of CY: Non-compact toric 3-folds

\[ (\phi_1, \ldots, \phi_{m+3}) - \{ \text{loc} \} \] / \( (\mathbb{C}^*)^m \)
Examples:

\[ (\phi_1, \phi_2, \phi_3, \phi_4) / C^* \]
weights \((1, 1, -1, -1)\)

\[ O(-1) + O(-1) \]
\[ \downarrow \]
\[ \mathbb{P}^1 \]

\[ (\phi_1, \phi_2, \phi_3, \phi_4, \ldots, \phi_{N+4}) / (C^*)^{N+1} \]
\[
\begin{pmatrix}
1 & -2 & 1 & 0 & \ldots \\
0 & 1 & -2 & 1 & 0 \\
0 & 0 & 1 & -2 & 1 \\
& & & & \ldots \\
& & & & 0 & -2 & 1 & 1
\end{pmatrix}
\]

\[ A_N \]
\[ \mathbb{P}^1 \]

resolved \(A_N\) singularity of \(\mathbb{C}^2 / \mathbb{Z}_{N+1}\)
over \(\mathbb{P}^1\):

superstring: CYXR^4

\[ \Rightarrow N=2 U(N) \]

rational curves \(\leftrightarrow\)

Geometric Engineering \(U(N)\) instantons on \(\mathbb{R}^4\)

Klemm
Katz
\[ \sigma \subset \text{holomorphic cycles} \]

\[ \sigma \subset \mathcal{CY}^3 \quad (\text{Neumann B.C.}) \]

\[ \sigma \subset \text{holomorphic Chern-Simons theory on } \mathcal{CY}^3 \]
\[ S = \int \Omega \wedge \text{tr}(\bar{A} \delta \bar{A} + \frac{2}{3} \bar{A}^3) \]

\[ \bar{A} : \text{hol. connection} \]

\[ \Omega : \text{holomorphic 3-form} \]

\[ \Omega \text{d version} \]

\[ \text{Matrix integrals} \]

\[ \text{example: } S = \text{tr}([X_i, X_j]X_k) \Omega_{ijk} \]

\[ \text{iji} = \]

\[ 123 \]

\[ + \ldots \text{ deformations} \]

\[ \text{Worldsheet diagrams } \rightarrow \]

\[ \text{Ribbon graphs of matrix model} \]
B/closed

\[ A_i, B_i \in H_3(\text{cr}) \]

\[ \{ \sum\Omega \sim t_i \}
\]

\[ \sum\Omega - F_i \]

\[ A_i \cap B_j = \delta_{ij} \]

\[ \rightarrow F_i = \partial_i F_0(t_i) \]

\[ F_0 : \text{genus 0-} \]

B model free energy:

Characterizes variation of Hodge Structures on CY.

\[ F = \sum_{g=0}^{9} F_g \frac{2^{9-2}}{5} \]

\[ t_5 : \text{string coupling constant} \]

"Quantum Kodaira-Spencer theory"

\[ F_i \sim \text{holomorphic Ray-Singer torsion} \]
$A \in \Omega^{\prime}(T)$
def of complex Structure

$\overline{\partial}_A = \overline{\partial} + A$

$\overline{\partial}_A^2 = 0 \rightarrow \overline{\partial}A + [A, A] = 0$

Quantum Kodaira–Spencer
theory has an action $S(A)$,
such that

$\frac{\delta S}{\delta A} = 0 \rightarrow \overline{\partial}A + [A, A] = 0$

Quantum Kodaira–Spencer
Mirror Symmetry

\[(C_{Y_1})_A = (C_{Y_2})_B\]

Kähler harder

Complex easier

Basic example:

\[\frac{1}{T^2}\]

\[A = R_1R_2 = -i2 = R_2R_1\]
Mirror for local CY
(toric):

\[ (\phi_1, \ldots, \phi_{N+3}) / \mathbb{C} \]

Weights

\[ (q_i, \ldots, q_i, q_{N+3}) \]

Mirror variables

\[ \text{Re}[Y_i] = |\phi_i|^2 \quad , \quad (\text{Im} Y_i) \rightarrow \arg(\phi_i) \]

\[ \sum_{j=1}^{N+3} q_i^j Y_j^i = t_i \]

\[ y_i = e^{-Y_i} \]

\[ \{ XZ - \sum_{i=1}^{N+3} (Y_i) = 0 \}
\[ \pi \sum_{j=1}^{N+3} q_i^j = e^{-t_i} \quad \mathbb{C} \times \mathbb{C} \times \mathbb{C}^2 \]

Riemann Surface

\[ F(0,0) \]
V.H.S. $\rightarrow x^2 - F(e^y, e^v) = 0$

$$
\int_{\Sigma} \left( \frac{d\Sigma}{x} \right) du dv = \int_{\text{Domain } H_1(\Sigma)} du dv \quad \partial D \equiv (F = 0)
$$

$0 (u, v)$

$(\Sigma F(u, v) = 0 : \Sigma)$

$F(u, v) = 0$

Solved by $\Sigma$

$A_2$ gauge theory
Mirror Symmetry
With Branes ($2E \neq \emptyset$)

Example:

Lagrangian Submanifold

$R^2 \times S^1$

$F(e, e^u e^v) = 0$

$S = \sum \int \lambda$ classical

$\lambda = u \, dv$

$\rightarrow$ non-trivial corrections
All A-model amplitudes on local toric CY can be reduced to

\[ Z = Z(U_0, V_0, W_0) \]

\[ U_0 = (e^{u_i}, e^{u_n}) \]
\[ V_0 = (e^{v_i}, e^{v_n}) \]
\[ W_0 = (e^{w_i}, e^{w_n}) \]

\[ R_i = \text{Rep of } U(\infty) \]
\[ R_1, R_2, R_3 \]

has been computed recently: Iqbal, Keshri, Aganagic, Klemm, Mari
cV., Dijkgraaf

\[ R_1 \]
\[ R_2 \]
\[ R_3 \]
There seems to be a new deep relation to classical 3D crystals!

\[ C_{R_1, R_2, R_3}(q) \quad q = e^{-g_s} \]

\[ C_{R_1, R_2, R_3} \propto \sum q^* \text{ boxes} \]

3D Young Tableaux
which asymptote \{ R_1, R_2, R_3 \}

Okounkov, Reshetikhin, V.

\( g_s \gg 1 \rightarrow \text{(Calabi-Yau} \rightarrow \text{3d lattice)} \)
A particular example of this:

\[ Z = \sum \frac{q^r}{r!} \]

3d Young tableau
with no fixed asymptotic restrictions

\[ Z = \exp \left( \sum \frac{C^3_{g-1}(H)}{g} \tilde{\mu}_g \lambda^2_{g-2} \right) \]

Gopakumar, V.

Faber, Pandharipande

\[ e^{-\lambda_s} = q \]

Mc Mahon's formula
Large $N$ dualities

Open $\leftrightarrow$ closed

$\mathcal{A}/TS^3$

$Z_{CS}^{S^3}(U(N)) +$ knots $= Z_{A-model}^{A-model}$

Open $\leftrightarrow$ closed

$O(-1) \otimes O(-1) \downarrow \mathbb{P}^1$

Ng$_s = t$ $\longleftrightarrow$ Gopakumar $V.$

Closed A-model

"Open A-model"
This duality can be lifted to a geometric statement for M-theory on $G_2$ manifolds:

\[ \text{Cone over } \left( \frac{S^3 \times S^3}{\mathbb{Z}_n} \right) \rightarrow \text{Cone over } (S^3 \times S^3) \]

Atiyah, Maldacena, V.
B-model

\[ \int \mathcal{D}\phi \, e^{-\frac{W(\phi)}{g_s}} \]

\[ W(\phi) = \sum_{r=0}^{n+1} \phi^r \alpha_r \]

Open

\[ xy + \omega^2 + \omega W'(z) + f_{n-1}(z) = 0 \]

Closed

\[ CY \text{ in } \mathbb{C}^4 \]

\[ (x, y, w, z) \]

Dijkgraaf, V.

Leading

\[ w = \langle \text{tr} \frac{1}{z - \phi} \rangle \]

\[ \Phi = (\Phi_1, \ldots, \Phi_N) \]

Large \( N \) = planar behavior. graphs

(standard techniques)

\[ \frac{1}{g_s} \int \mathcal{W} \, dz = N_i \]

\[ \mathcal{A}_i \]
Connections to \( N=1 \) SYM \\
\( d=4 \) \\

Dijkgraaf, V.

As mentioned

B-model open version in 0-d:

\[
S D\phi \in W(\phi) \quad \text{matrix model}
\]

\[
\text{embed: } \mathbb{R}^4 \times \mathbb{C}^4 \to \mathbb{C}^3
\]

\( N=1 \) SYM, \( d=4 \) with \( \phi \) matter, \( W(\phi) \) superpotential.
$W'(\phi_i) = 0 \Rightarrow$
certain vacua
$(M_1, \ldots, M_n)$

$\rightarrow$ Planar graphs $(M_1, \ldots, M_n)$

$F_0 \left( S_1, \ldots, S_n \right)$

$S_i = M_i g_s$

Bershadsky et al.

$W(S_i) = N_i \frac{\partial F_0}{\partial S_i}$

$\text{space-time super potential}$

$S_i = \text{"glueball fields"}$

$W'(S_i) = 0 \Rightarrow \text{Vacuum geometry!}$

including instantons!
This leads to a Perturbative window into Non-perturbative (i.e. instanton) physics.

Example:

N=1 theory with 3-adjoints $X,Y,Z$

with $W(X,Y,Z) = t \{X,Y\} Z$

$\cong (N=4$ Yang-Mills$)$

\[ \int D X D Y D Z \ e^{\frac{-t}{g_s} \{X,Y\} Z + m (x^2 + y^2 + z^2)} \]

$\Rightarrow W(S) = N \frac{\delta F}{\delta S} - \frac{1}{2} S^2$

planar graphs

$\frac{\delta W}{\delta S} = 0 \Rightarrow W|_{\min} = m^3 E_2(\hat{t})$

$E_2(\hat{t}) \approx \sum c_i (m^4)^i$
Montonen-Olive duality $\hat{\tau} \to -\frac{1}{\hat{\tau}}$ 

$N=4$ YM

Modularity of $E_2(\hat{\tau})$

Topological strings $\rightarrow$ strong coupling dualities in gauge theories.
A highly non-trivial but satisfying picture

Unity of Topological Field theories!