**$A_\infty$-structures**

This talk introduces $A_\infty$-algebras. I want to keep things very concrete, stating some of the main constructions explicitly (as opposed to, let’s say, describing them as features of the model category of $A_\infty$-algebras). The main point is how much information is actually contained in the higher order products. There are no strict prerequisites for giving this talk, but background in rational homotopy theory may be helpful.

**Plan**

*Definitions.* It will be enough for us to work over a field, for instance $\mathbb{C}$. Define the notion of $A_\infty$-algebra, homomorphism of $A_\infty$-algebras, and quasi-isomorphism. The main theorem is that any quasi-isomorphism is invertible (proof will follow later on). Basic references are [Keller, lectures on $A_\infty$-algebras and modules], [Seidel, Fukaya categories and Picard-Lefschetz theory, Chapter 1], [Lefevre-Hasegawa, sur les $A_\infty$-catégories, it’s a thesis but I have a copy].

At some point, one needs to explain that an $A_\infty$-structure yields triple Massey products on the cohomology level. This is nice because one can see the $m_3$ term appearing explicitly in the formula for the Massey product (it is also a kind of baby version of the next topic).

*Perturbation theory.* Let’s see the proof of the Perturbation Lemma. I strongly suggest following the explicit sum-over-trees method given in [Kontsevich-Soibelman, Homological mirror symmetry and torus fibrations]. Other references are [Markl, Transferring $A_\infty$ (strongly homotopy associative) structures] or [Merkulov, Strongly homotopy algebra of a Kähler manifold].

The Perturbation Lemma implies the theorem above on invertibility of quasi-isomorphisms. Compare the corresponding argument for the $L_\infty$-case in [Kontsevich, Deformation quantization of Poisson manifolds].

*Examples.* As an application of the Perturbation Lemma, the cohomology of any space has an $A_\infty$-structure (unique up to isomorphism, but not unique on the nose). A classical theorem of Deligne-Griffiths-Morgan-Sullivan says that for compact Kähler manifolds, the $A_\infty$-structure is in fact trivial (the original paper does not use $A_\infty$-terminology, but proves an equivalent result; for an $A_\infty$ viewpoint, see the more recent [Merkulov, op. cit.], which unfortunately doesn’t quite get to the theorem).

Another topological example which I quite like is the following one. Let $M$ be a homology three-sphere. We consider the moduli space of flat $GL(N)$ connections over $M$. The ordinary cohomology describes the Zariski tangent space to that moduli space (at the trivial connection), but the $A_\infty$-structure on it describes the entire formal neighbourhood of the trivial connection. This involves rewriting the Maurer-Cartan equation on cochains, $da + a^2 = 0$, in terms...
of the $A_\infty$-structure as generalized Maurer-Cartan equation on cohomology [no explicit reference, but old papers of Goldman and Millson on moduli spaces of flat connections will be helpful].

Dependencies. None! However, we have a talk later on on Hochschild cohomology, which will complement this one in some respects.