This talk introduces derived categories. To prevent boredom, I want to stay away from a too classical viewpoint, and emphasize the intrinsic nature of exact triangles on the $A_\infty$ level. Much of this is due to Kontsevich. A basic reference is [Keller, lectures on $A_\infty$-algebras and modules]. It would still be good if the speaker is acquainted with classical derived categories, or even better with [Bondal-Kapranov, framed triangulated categories]. Most of the material can be found in some form in [Seidel, Fukaya categories and Picard-Lefschetz theory, Chapter I].

Plan

$A_\infty$-categories. Reminder of $A_\infty$-categories, $A_\infty$-functors, and the invertibility of quasi-equivalences. Take the standard category $T_2$ with two objects and one isomorphism in either direction. Then, two objects in an arbitrary $A_\infty$-category $\mathcal{A}$ are said to be isomorphic if there is an $A_\infty$-functor $T_2 \to \mathcal{A}$ mapping the two model objects to them.

Exact triangles. Generalizing the previous example, there is a special $A_\infty$-category with three objects, the Kontsevich triangle category $T_3$, described in [Seidel op. cit., Section 3g]. One can think of this as an $A_\infty$-deformation of $\Lambda(\mathbb{C})/\mathbb{Z}/3$, where $\mathbb{Z}/3$ acts on $\mathbb{C}$ by rotation. In any case, exact triangles in $\mathcal{A}$ are defined as images of the standard triangle under $A_\infty$-functors $T_3 \to \mathcal{A}$. One should then prove some standard properties: the fact that all $A_\infty$-functors preserve exact triangles is obvious from this point of view; however, one has to prove that an exact triangle induces long exact sequences of $Hom$ spaces on the left and right; and more challengingly, if one changes the arrows in an exact triangle to cohomologous ones, the resulting triangle is still exact. Some of this may be postponed to later, when we have twisted complexes and such.

Triangulated $A_\infty$-categories. An $A_\infty$-category is called triangulated if it is closed under up and down shifts, and if every morphism can be completed to an exact triangle. The first example are categories of $A_\infty$-modules, and mapping cones. If the speaker is feeling ambitious, this would be the moment to introduce an $A_\infty$-version of the notion of strong generators, and to prove that an arbitrary triangulated $A_\infty$-category admitting suitable colimits, and having a strong generator, is quasi-equivalent to the category of modules over the endomorphism ring of that generator. However, I don’t know a reference for this (the corresponding result for dg categories is in [Keller, math.KT/0601185], and I assume one could derive the $A_\infty$-case from that if necessary; however, it may be too much trouble to define what one actually means by admitting suitable colimits).

Twisted complexes. For $A_\infty$-categories, this was first introduced by [Kontsevich, Homological algebra of mirror symmetry], even if his definition is a little
clumsy. One should define them, mention that the resulting $A_\infty$-categories of twisted complexes are triangulated, define the notion of generators and of the triangulated envelope, which then leads to the definition of derived category of an $A_\infty$-category. As usual, we also need a nice example. I feel that the directed $A_n$ quivers give a nice example, since there one can see various indecomposable objects in the derived category as twisted complexes (this is an old story, going back to [Bernstein-Gelfand-Ponomarev] on reflection functors).

Notes

Kontsevich’s $T_3$ has a generalization $T_n$, which is in fact related to the derived category of the $A_n$ quiver. There is a nice explanation of the octahedral axiom in this context [Kontsevich, unpublished]. These $A_\infty$-categories can also be viewed as deformations of $\Lambda(\mathbb{C}) \rtimes \mathbb{Z}/n$, where $\mathbb{Z}/n$ acts on $\mathbb{C}$ by $\exp(2\pi i/n)$.

Another topic which may or may not fit into this lecture is idempotent completion (split-closure) and the associated notion of split-generator. However, since that is not particularly deep, we may fit it in anywhere later when it’s needed.

Dependencies

This relies on: the basic talk on $A_\infty$-structures.