Symmetric monoidal ∞-categories and the stable motivic ∞-category

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18 Mar 2014

Symmetric monoidal ∞-categories

In the 1-categorical world, we have both strict and lax symmetric monoidal categories. We clearly want to generalize the latter, but the classical definition involves some weird axioms (e.g. an associativity pentagon); this would be prohibitively messy for higher categories. So we should find a new definition of symmetric monoidal 1-categories.

Construction. w Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. Define a category $\mathcal{C}^{\otimes}$ as follows:

- Objects are collections $(A_i)_{i \in I}$ of objects in $\mathcal{C}$, indexed by a finite set;
- Morphisms from $(A_i)_{i \in I}$ to $(B_j)_{j \in J}$ consist of a map $\sigma : I^+ \to J^+$ in Fin_*, the category of finite pointed sets, together with a map

  $$\bigotimes_{\sigma(i)=j} A_i \to B_j$$

  for each $j \in J$.

Remark. —

- The category of finite pointed sets is equivalent to the category of finite sets and partially defined maps, via the "add a disjoint basepoint" functor.
- There is a functor $\mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ taking each object to its indexing set and each morphism to the first part of its defining data.
- This functor is co-cartesian: given a morphism $\sigma : \langle n \rangle \to \langle m \rangle$ in Fin_* and a lift $A = (A_k)_{k \in \langle n \rangle}$ of the domain, we define a co-cartesian lift by letting $B_j = \otimes_{\sigma(i)=j} A_i$, and defining the morphism $A \to B$ by identity morphisms in $\mathcal{C}$.
- We have that the fiber $\mathcal{C}^{\otimes}_{\langle n \rangle}$ is equivalent to n copies of $\mathcal{C}$. More precisely, define $i_k : \langle n \rangle \to \langle 1 \rangle$ to be the morphism mapping only k to 1. Then the product of co-cartesian lifts

  $$i_1 \times \ldots \times i_n : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\otimes}_{\langle 1 \rangle} \times \ldots \times \mathcal{C}^{\otimes}_{\langle 1 \rangle}$$

is an equivalence.

- In particular, we can identify $\mathcal{C}$ with $\mathcal{C}^{\otimes}_{\langle 1 \rangle}$.

We can recover the symmetric monoidal structure purely from this functor $\mathcal{C}^{\otimes} \to \mathsf{Fin}_*$. For instance, we can form a tensor product of objects by choosing an inverse $\mathcal{C}^{\otimes}_{\langle 1 \rangle} \times \mathcal{C}^{\otimes}_{\langle 1 \rangle} \to \mathcal{C}^{\otimes}_{\langle 2 \rangle}$ to the equivalence above, and then forming

$$\mathcal{C}^{\otimes}_{\langle 1 \rangle} \times \mathcal{C}^{\otimes}_{\langle 1 \rangle} \to \mathcal{C}^{\otimes}_{\langle 2 \rangle} \xrightarrow{d_1} \mathcal{C}^{\otimes}_{\langle 1 \rangle}.$$

Note that this depends on a choice of inverse, so there are various possible tensor products (all equivalent). We recover the monoidal unit by lifting $\langle 0 \rangle \to \langle 1 \rangle$.

This is an adequate characterization of lax symmetric monoidal categories, and prompts the following definition:

Definition. A **symmetric monoidal ∞-category** is a co-cartesian fibration $\mathcal{C}^{\otimes} \to \mathrm{N}\mathsf{Fin}_*$ such that

$$i_1 \times \ldots \times i_n : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\otimes}_{\langle 1 \rangle} \times \ldots \times \mathcal{C}^{\otimes}_{\langle 1 \rangle}$$

is an equivalence.

Example. If an ∞-category has all finite products, then they yield a symmetric monoidal structure. See Lurie, *Higher Algebra*, section 2.4; the proof is doable but not trivial, using constructions to do with the nerve of the posets of subsets of some set, much like the simplicial nerve.

We should define symmetric monoidal functors, and here we still have notions of strict and lax functors.

Definition. A morphism $\langle n \rangle \to \langle m \rangle$ of Fin_* is **inert** if each element of the codomain has preimage of size 1. So these are the bijections from a subset of $\langle n \rangle$ to all of $\langle m \rangle$.

Definition. —

- A **strict symmetric monoidal functor** $\mathcal{C} \to \mathcal{D}$ is a commutative triangle

$$\begin{array}{ccc} \mathcal{C}^{\otimes} & \longrightarrow & \mathcal{D}^{\otimes} \\ & \searrow \quad \swarrow & \\ & \mathrm{N}\mathsf{Fin}_* & \end{array}$$

  which preserves cocartesian morphisms.

- A **lax symmetric monoidal functor** is only required to preserve cocartesian morphisms over inert morphisms of Fin_*.

- We let SymMon denote the presentable ∞-category of symmetric monoidal categories with strict symmetric monoidal functors. We won't construct this.

Definition. A **commutative algebra object** in a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is a lax symmetric monoidal functor $\mathsf{NFin}_* \to \mathcal{C}^{\otimes}$.

So the image of $\langle 1 \rangle \in \mathsf{NFin}_*$ gives an object A of $\mathcal{C}$. From $d_1 : \langle 2 \rangle \to \langle 1 \rangle$ in NFin_*, we obtain, up to equivalences, a map $A \otimes A \to A$. This definition should now seem plausible.

Fact. Symmetric monoidal categories are equivalent to the commutative algebra objects in Cat_∞ (with the monoidal structure on Cat_∞ given by products).

Given the theory of ∞-operads, we could also define symmetric monoidal categories as the algebras in Cat_∞ of an ∞-operad defining a commutative operation. We also have a good theory of modules over algebras over an operad, which we will refer to briefly later.

Inversion

Inversion of objects in a symmetric monoidal category is not so unfamiliar: the passage from stable to unstable homotopy is the inversion of S^1 under the smash product. This suits our need to pass to a stable motivic ∞-category. Let us more precisely state what we want.

Definition. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. An object $A \in \mathcal{C}$ is **invertible** if there exists $B \in \mathcal{C}$ with $A \otimes B \simeq 1 (\simeq B \otimes A)$. Equivalently, we could ask for the map $\mathcal{C} \xrightarrow{-\otimes A} \mathcal{C}$ to be an equivalence.

So given $\mathcal{C}$ and any object A, we want some (strict) symmetric monoidal functor $\mathcal{C} \to \mathcal{D}$ taking A to an invertible object, and we want one which is initial with this property. Let $\mathsf{SymMon}^A_{\mathcal{C}/}$ denote the category of such functors inverting A; so we want to see that $\mathsf{SymMon}^A_{\mathcal{C}/}$ has an initial object.

Proposition. *The inclusion* $\mathsf{SymMon}^A_{\mathcal{C}/} \hookrightarrow \mathsf{SymMon}_{\mathcal{C}/}$ *is a* **reflective subcategory***, i.e. the right adjoint to an accessible localization.*

Proof. We want to identify the left hand side as the subcategory on S-local objects, for some collection of morphisms S. So take the singleton

$$S = \{\mathrm{Free}_{\mathcal{C}}(\mathcal{C}) \xrightarrow{-\otimes A} \mathrm{Free}_{\mathcal{C}}(\mathcal{C})\},$$

where we form $\mathrm{Free}_{\mathcal{C}}(\mathcal{C})$ by taking the free $\mathcal{C}$-algebra on the $\mathcal{C}$-module $\mathcal{C}$. (We haven't defined this thoroughly, but we'll use its straightforward universal property.)

Now an object Z is S-local iff

$$\hom(\mathrm{Free}_{\mathcal{C}}(\mathcal{C}), Z) \xrightarrow{(-\otimes A)^*} \hom(\mathrm{Free}_{\mathcal{C}}(\mathcal{C}), Z)$$

is an equivalence. By the universal property of the free object, this is precisely the condition that $Z \xrightarrow{-\otimes A} Z$ is an equivalence, or that A is invertible in Z, as desired.

Now the general theory of accessible localizations applies: the inclusion of the S-local objects into a presentable category is always reflective. $\square$

So now we take the left adjoint $\mathcal{L} : \mathsf{SymMon}_{\mathcal{C}/} \to \mathsf{SymMon}^{A}_{\mathcal{C}/}$. This must preserve colimits, so it takes the initial object $\mathrm{id}_{\mathcal{C}}$ to an initial object of $\mathsf{SymMon}^{A}_{\mathcal{C}/}$, as desired.

The stable motivic ∞-category

Last talk, Jay constructed $\mathcal{H}(S)$, the unstable motivic ∞-category. This has products, and thereby becomes symmetric monoidal. We can modify this into a symmetric monoidal structure, the smash product, on the pointed unstable category $\mathcal{H}_*(S) = \mathcal{H}(S)_{*/}$. So we want to stabilize this by inverting a sphere.

Question. What sphere?

There are many candidate spheres in this category:

- S^1, obtained as a pushout of points

$$\begin{array}{ccc} * \coprod * & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & S^1 \end{array}$$

- $\mathbb{G}_m = \mathbb{A}^1 \smallsetminus 0$,
- $\mathbb{P}^1$,
- $\mathbb{A}^1/\mathbb{G}_m$.

Notice that any S-scheme yields a functor of points, which is a presheaf on $\mathrm{Sm}_{/S}$, and thereby becomes an object of the motivic category.

We can define the general motivic sphere $S^{p,q} = (S^1)^{\wedge p} \wedge (\mathbb{G}_m)^{\wedge q}$. (This is not the only choice of how to index the bigrading.) Then for example we have $(\mathbb{P}^1, \infty) \simeq S^1 \wedge \mathbb{G}_m$, as the following elementary Nisnevich square becomes a pushout in the motivic category:

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

and $\mathbb{A}^1 \simeq *$, so that this pushout is a suspension diagram.

In order to obtain a stable ∞-category, and thereby a triangulated homotopy category, we must invert at least S^1. By doing so, we obtain the category $\mathcal{SH}_{S^1}(S)$. But this will be inadequate for most purposes, such as representing various cohomology theories. We ought to invert all of the spheres, or equivalently just $\mathbb{P}^1 \simeq S^{1,1}$.

Definition. The **stable motivic ∞-category** $\mathcal{SH}(S)$ is the formal inversion of $\mathbb{P}^1$ in $\mathcal{H}_*(S)$.

This now has the universal property:

Proposition. *For any pointed presentable symmetric monoidal ∞-category $\mathcal{D}$, the map*

$$\mathrm{Fun}^{\otimes,L}(\mathcal{SH}(S),\mathcal{D}) \to \mathrm{Fun}^{\otimes}(\mathrm{N}(\mathrm{Sm}_{/S}),\mathcal{D})$$

is fully faithful, with image consisting of those functors satisfying Nisnevich descent, $\mathbb{A}^1$-invariance, and taking $\mathbb{P}^1$ to an invertible object.

For basepoint reasons, we should strictly say the cofiber of the point at infinity, $* \xrightarrow{\infty} \mathbb{P}^1$, rather than $\mathbb{P}^1$.