Chromatic homotopy theory at height 1 and the image of J

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April 23, 2013

Key players at height 1

Formal group law:
Let $F_m(x, y)$ be the $p$-typification of the multiplicative formal group law $x + y + xy$ over $F_p$. Then the $p$-series of $F_m$ is

$$[p]_{F_m}(x) = x^p$$

Thus, $F_m$ is exactly the height 1 Honda formal group law.

Morava E-theory

In general, we have that $E(k, \Gamma) = \mathbb{W}k[[u_1, \ldots, u_{n-1}]]$

In this case,

$$\mathbb{W}F_p = \mathbb{Z}_p$$

and

$$E(F_p, F_m) = \mathbb{Z}_p$$

Adjoining an invertible class in degree -2 to make this into an even periodic theory, we have that the first Morava E-theory $E_1$ has coefficients $\mathbb{Z}_p[u^{\pm 1}]$. Furthermore, we may take $F$ as a universal deformation of itself. Turning it into a degree -2 formal group law, we have

$$F(x, y) = u^{-1}F(ux, uy)$$

$E_1$ is a model for $p$-complete complex $K$-theory (it has the same coefficients and the same formal group law).

Morava stabilizer group:

We are interested in the group of endomorphisms of the multiplicative formal group law, $F$. First, note that it must contain $\mathbb{Z}$: given an integer $n \in \mathbb{Z}$, we send it to the $n$-series $[n]_F(x)$. We may extend this to $\mathbb{Z}_p$, since a $p$-adically convergent sequence of integers
$n_1, n_2, \ldots$ gives a $p$-adically convergent sequence of power series $[n_1](x), [n_2](x), \ldots$ (this requires checking $[p^r](x) \mod x^{pr+1}$). It turns out that there are no other endomorphisms, i.e. $\text{End}(F) \cong \mathbb{Z}_p$.

$S_1$, the group of automorphisms of $F$, is the group of units in the $p$-adics. The reduction mod $p$ map $\mathbb{Z}_p^\times \to \mathbb{F}_p^\times$ sits in a short exact sequence

$$1 \to 1 + p\mathbb{Z}_p \to \mathbb{Z}_p^\times \to \mathbb{F}_p^\times$$

Thinking of $\mathbb{F}_p^\times$ as the group $\mu_{p-1}$ of $(p-1)$st roots of unity over $\mathbb{F}_p$, we may use Hensel’s Lemma to construct a splitting $\mu_{p-1} \to \mathbb{Z}_p^\times$ so that $\mathbb{Z}_p^\times \cong \mu_{p-1} \times (1 + p\mathbb{Z}_p)$.

For odd primes, the above is topologically cyclic, generated by any element $g = (\zeta, \alpha)$ such that $\zeta$ is a primitive $(p-1)$st root of unity and $\alpha \notin 1 + p^2\mathbb{Z}_p$. For $p = 2$, we have $\mathbb{Z}_2^\times \cong \{\pm 1\} \times (1 + 4\mathbb{Z}_2)$ and while $\mathbb{Z}_2^\times$ is not topologically cyclic, $1 + 4\mathbb{Z}_2$ is.

$S_1$ acts on the homology theory $E_1$ as follows. Given an automorphism $g(x)$ of the multiplicative formal group law over $\mathbb{F}_p$, we lift the coefficients to $\mathbb{Z}_p$ and adjust for the grading to get $\tilde{g}(x) = u^{-1}g(ux)$. The induced map $\psi : \mathbb{Z}_p = E(\mathbb{F}_p, F) \to E(\mathbb{F}_p, F) = \mathbb{Z}_p$ must be the identity, and so $\psi^*F = F$. We extend $\psi$ to $\mathbb{Z}_p[u^{\pm 1}]$ by defining $\psi(u^{-1}) = g'(0)u^{-1}$.

This is the action of $S_1$ on the coefficients of $E_1$. When $g(x) = [n]_F(x) = (1 + x)^n - 1$, then $\tilde{g}(x) = u^{-1}((1 + ux)^n - 1)$, $g'(0) = n$, and

$$\psi(u) = nu$$

The action on the homology theory $E_1$ is given by applying Landweber exactness. $\psi : \mathbb{Z}_p[u^{\pm 1}] \to \mathbb{Z}_p[u^{\pm 1}]$ is a map of $MU_*$-modules and so we have an automorphism of $E_1 = \mathbb{Z}_p[u^{\pm 1}] \otimes_{MU_*} MU_*(-)$.

**Defining the image of J**

Let $\mathcal{H}(n)$ denote the monoid of homotopy self-equivalences of $S^n$ that preserve the basepoint. It sits inside $\Omega^n S^n$ as the union of two components. There is an obvious map $O(n) \to \mathcal{H}(n)$ (There is also a map from $U(n)$ to $\mathcal{H}(2n)$ which factors through $O(2n)$). The composition

$$O(n) \to \mathcal{H}(n) \to \Omega^n S^n$$
induces
\[ \pi_i(O(n)) \to \pi_i(\Omega^n S^n) = \pi_{n+i} S^n \]
and we can check that these maps commute with the maps \( O(n) \to O(n+1) \) and \( H(n) \to H(n+1) \) to yield a map of colimits
\[ \phi : O \to H \to \Omega^\infty \Sigma^\infty S^0 \]
and a map of stable homotopy groups
\[ \pi_i O \to \pi_i^s \]
We call this map the \( J \)-homomorphism and denote its image by \( J(S^i) \).
The map \( O(n) \to H(n) \) sits in a fiber sequence
\[ O(n) \to H(n) \to H(n)/O(n) \to BO(n) \to B H(n) \]
Notice that \( BO(n) \to B H(n) \) is the map that classifies the underlying spherical fibration of the universal bundle over \( BO(n) \).

**Stable Adams operations**

Recall that the Adams operations \( \psi^k : K(X) \to K(X) \) are the unique natural ring homomorphisms such that \( \psi^k(L) = L^k \) whenever \( L \) is a line bundle. They are unstable in the sense that the diagram
\[
\begin{array}{ccc}
\Sigma^2 BU & \xrightarrow{B} & BU \\
1 \land \psi^k \downarrow & & \downarrow \psi^k \\
\Sigma^2 BU & \xrightarrow{B} & BU
\end{array}
\]
does not commute. If we invert \( k \), we can fix this by defining \( \tilde{\psi}^k \) on the \( 2^n \)th space of \( KU \) by \( \tilde{\psi}^k = \frac{\psi^k}{k^\infty} \). It maps to \( KU[\frac{1}{k}] \). Since \( \psi^k \) acts on the Bott class \( \beta \in \pi_2(BU) \) by \( \psi^k(\beta) = k \beta \) and the map \( \Sigma^2 KU_0 \to KU_2 \) is just multiplication by \( \beta \). Our definition of \( \tilde{\psi}^k \) adjusts for this.

If we complete at a prime \( p \), then \( \tilde{\psi}^k \) is defined for \( k \) coprime to \( p \) and from here on we drop the tilde and refer to these stable Adams operations as \( \psi^k \). So \( \mathbb{Z} \) sits inside \([K_p, K_p] \). One can show that \([K_p, K_p] \) is complete, so that the Adams operations extend to \( \mathbb{Z}_p \).

This actually turns out to be an isomorphism.

Notice that the action of the stable Adams operations on
\[ \pi_* KU_p = \mathbb{Z}_p[\beta^\pm 1] \]
is exactly the action of the Morava stabilizer group on \( E_1 \).

**The Adams conjecture**

**Adams Conjecture:** If \( k \in \mathbb{N} \), then for any \( x \in K(X) \), we have \( k^n(\psi^k(x) - x) = 0 \) in the image of \( J \) for some \( n \gg 0 \).

This gives us an upper bound on the image of \( J \).
The image of $J$ completed at $p$

If we complete at a prime $p$, the Adams conjecture implies that the composition of the map $1 - \psi^k$ with $BU_p \to B\mathcal{H}_p$ is nullhomotopic whenever $k$ is coprime to $p$. This induces a map $\text{hofib}(1 - \psi^k) \to \mathcal{H}$ such that the following diagram commutes

$$
\begin{array}{cccc}
U_p & \longrightarrow & \text{hofib}(1 - \psi^k) & \longrightarrow & BU_p \\
\downarrow & & \downarrow & & \downarrow \\
U_p & \longrightarrow & \mathcal{H}_p & \longrightarrow & \mathcal{H}_p/U_p \\
\downarrow & & \downarrow & & \downarrow \\
& & BU_p & & BU_p
\end{array}
$$

Thus, we have shown that the Adams conjecture implies that the $J$ homomorphism factors through the homotopy of the fiber of $1 - \psi^k$. If $p$ is odd, let $g$ be a generator of $\mathbb{Z}_p^\times$. Then in fact $\text{hofib}(1 - \psi^g)$ is a split summand of $\mathcal{H}_p$. We will come back to this later. For now, let’s assume

**Theorem:** The map $\pi_n(\text{hofib}(1 - \psi^g)) \to \pi_nB\mathcal{H}_p = \pi_nS^0_p$ is the inclusion of a split summand of $\pi_nS^0_p$ for $n \geq 0$.

**hofib($1 - \psi^g$) and $L_{K(1)}S$**

The theorem of Devinatz-Hopkins that $L_{K(n)}S^0 = E_n^{hS_n}$ in this case says that $L_{K(1)}S^0 = K^{h\mathbb{Z}_p^\times}$

**Proposition:** Let $g$ be a topological generator of $\mathbb{Z}_p$. Then $K^{h\mathbb{Z}_p^\times} = \text{hofib}(1 - \psi^g)$.

**Proof:** Consider the diagram

$$
\begin{array}{cccc}
K_p^\mathbb{Z} & \longrightarrow & K_p & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
K_p & \longrightarrow & K_p \times K_p & \longrightarrow & K_p
\end{array}
$$

Both squares are pullbacks. This implies that the fibers of the horizontal compositions are equivalent. That is, $\text{hofib}(1 - \psi^g) = K^{h\mathbb{Z}_p}$. It remains to show that $K^{h\mathbb{Z}_p} = K^{h\mathbb{Z}_p^\times}$. To see this we use the homotopy fixed point spectral sequence. There is a map $K^{h\mathbb{Z}_p} \to K^{h\mathbb{Z}_p^\times}$ given by inclusion of fixed points. It induces a map of homotopy fixed point spectral sequences, and

$$
H^*_c(\mathbb{Z}_p^\times, \pi_*(K_p)) \to H^*_c(\mathbb{Z}, \pi_*(K_p))
$$

is an isomorphism of $E_2$-terms. This can be seen by computing both of them.
Computing $\pi_n L_{K(1)} S$

Since $L_{K(1)} S = \text{hofib}(1 - \psi^g)$, we may use the long exact sequence of the fibration

$$L_{K(1)} S \rightarrow K_p \rightarrow K_p$$

to compute the homotopy of $L_{K(1)} S$. Recall that $g$ is chosen to be a generator of $\mathbb{Z}_p^\times = \mu_{p-1} \times (1 + p\mathbb{Z}_p)$ so that $g = (\zeta, y)$ where $p$ divides $g - 1$ but $p^2$ does not. Firstly, since $\psi^g$ acts on $\pi_0 K_p = \mathbb{Z}_p$ by the identity, $1 - \psi^g$ vanishes on $\pi_0$ and since $\pi_2 K_p = 0$, we have

$$\pi_0 L_{K(1)} S \cong \mathbb{Z}_p$$

and

$$\pi_{-1} L_{K(1)} S \cong \mathbb{Z}_p$$

On $\pi_{2k} K_p$, $\psi^g$ acts by $g^k$. Thus, $1 - \psi^g$ is injective for $k \neq 0$ and so

$$\pi_{2k} L_{K(1)} S = 0$$

and

$$\pi_{2k-1} L_{K(1)} S \cong \mathbb{Z}_p / (1 - g^k)$$

Now, if $p - 1$ does not divide $k$, then $g^k - 1$ is a unit mod $p$ and $\pi_{2k-1} L_{K(1)} S = 0$. If $k = (p - 1)m$, then $g^k = (g^{p-1})^m$, and $g^{p-1}$ topologically generates $1 + p\mathbb{Z}_p$. If $m = p^r l$ where $l$ is coprime to $p$, then $(g^{p-1})^m = ((g^{p-1})^{p^r})^m$ topologically generates the cyclic subgroup $1 + p^{r+1} \mathbb{Z}_p$ so that $1 - g^k$ generates $p^{r+1} \mathbb{Z}_p$ topologically. Thus, if $k = (p - 1)p^r l$, then

$$\pi_{2k-1} = \mathbb{Z} / p^{r+1}$$

That is,

$$\pi_n L_{K(1)} S = \begin{cases} 
\mathbb{Z}_p & : n = 0, -1 \\
\mathbb{Z} / p^{r+1} \mathbb{Z} & : n + 1 = 2(p - 1)p^r l \neq 0 \text{ mod } p \\
0 & : \text{otherwise}
\end{cases}$$

Aside: Bernoulli numbers

The Bernoulli numbers $\beta_t$ are given by the power series of the function $x / (e^x - 1)$:

$$\frac{x}{e^x - 1} = \sum_{t=0}^{\infty} \beta_t \frac{x^t}{t!}$$

Since $\frac{x}{e^x - 1} - 1 + \frac{x}{2}$ is an even function, $\beta_{2t+1} = 0$ for $t > 0$. Also, $\beta_1 = -\frac{1}{2}$.

This definition of Bernoulli numbers will come up in the lower bound of the image of $J$. What we will really be interested in are the denominators of $\frac{\beta_{2t}}{15}$ when the fraction is expressed in lowest terms. Call this $m(2s)$. Adams describes $m(2s)$ by giving its $p$-adic evaluation:

**Proposition:** For $p$ odd, $\nu_p(m(t)) = 1 + \nu_p(t)$ if $(p - 1)$ divides $t$ and is zero otherwise. For $p = 2$, $\nu_2(m(t)) = 2 + \nu_2(t)$ if $t$ is even and 1 otherwise.

Notice that the order of $\pi_n L_{K(1)} S$ is exactly $\nu_p(\frac{n+1}{2})$, that is, the denominator of $\beta_{(n+1)/2} / (n+1)$. 5
Computing $\pi_* L_{E(1)} S$

To compute $\pi_* L_{E(1)} S$, we use the pullback square

$$
\begin{array}{ccc}
L_{E(1)} S & \longrightarrow & L_{K(1)} S \\
\downarrow & & \downarrow \\
L_{E(0)} S & \longrightarrow & L_{E(0)} L_{K(1)} S
\end{array}
$$

which gives a long exact sequence in homotopy

$$
\cdots \rightarrow \pi_{n+1} L_{E(0)} L_{K(1)} S \rightarrow \pi_n L_{E(1)} S \rightarrow \pi_n L_{K(1)} S \oplus \pi_n L_{E(0)} S \rightarrow \pi_n L_{E(0)} L_{K(1)} S \rightarrow \cdots
$$

Recall that $L_{E(0)} S \simeq H \mathbb{Q} \simeq S \mathbb{Q}$ (where the right hand side is the rational Eilenberg-Moore spectrum). There is a universal coefficients theorem for $\pi_* SG$

$$
0 \rightarrow G \otimes \pi_* X \rightarrow \pi_* (SG \wedge X) \rightarrow \text{Tor}(G, \pi_{*-1} X) \rightarrow 0
$$

which for $G = \mathbb{Q}$ implies that

$$
\pi_* L_{E(0)} L_{K(1)} S \cong \pi_* (L_{E(0)} S \wedge L_{K(1)} S) \cong \mathbb{Q} \otimes \pi_* (L_{K(1)} S)
$$

Thus, $\pi_n (L_{E(0)} L_{K(1)} S) \cong \mathbb{Q}_p$ for $n = 0, -1$ and is zero otherwise.

Then for $n \neq 0, -1, -2$ we have $\pi_n L_{E(1)} S \cong \pi_n L_{K(1)} S$. For the remaining groups, we have the exact sequence

$$
0 \rightarrow \pi_0 L_{E(1)} S \rightarrow \mathbb{Z}_p \oplus \mathbb{Q} \rightarrow \mathbb{Q}_p \rightarrow \pi_{-1} L_{E(1)} S \rightarrow \mathbb{Z}_p \rightarrow \mathbb{Q}_p \rightarrow \pi_{-2} L_{E(1)} S \rightarrow 0
$$

It follows that

$$
\pi_n L_{E(1)} S = \begin{cases} 
\mathbb{Z} & : n = 0 \\
\mathbb{Q}_p/\mathbb{Z}_p & : n = -2 \\
\mathbb{Z}/p^{r+1} \mathbb{Z} & : n + 1 = 2(p - 1)p^r l \not\equiv 0 \mod p \\
0 & : \text{otherwise}
\end{cases}
$$

Adams’ lower bound on the image of $J$

The $e$-invariant

Adams computed a lower bound on the image of $J$ and showed that it is the same as the upper bound. The computation consists of defining a homomorphism

$$
e : \pi_k^s \rightarrow \mathbb{Q}/\mathbb{Z}
$$

such that the composition

$$
\pi_{2k-1} U(n) \xrightarrow{J} \pi_{2n+2k-1} S^{2n} \xrightarrow{e} \mathbb{Q}/\mathbb{Z}
$$

when evaluated on a generator of $\pi_{2k-1} U(n)$ has denominator $m(k)$. 

Given a map \( g : S^{2m-1} \to S^{2n} \), let \( C_g \) denote the cofiber. Then we have a short exact sequence in \( K \)-theory

\[
0 \to \tilde{K}(S^{2m}) \to \tilde{K}(C_g) \to \tilde{K}(S^{2n}) \to 0
\]

Applying the Chern character, we have a homomorphism of short exact sequences

\[
\begin{array}{cccc}
0 & \longrightarrow & \tilde{K}(S^{2m}) & \longrightarrow & \tilde{K}(C_g) & \longrightarrow & \tilde{K}(S^{2n}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{H}^*(S^{2m}; \mathbb{Q}) & \longrightarrow & \tilde{H}^*(C_g; \mathbb{Q}) & \longrightarrow & \tilde{H}^*(S^{2n}; \mathbb{Q}) & \longrightarrow & 0
\end{array}
\]

Let \( \alpha, \beta \) denote elements of \( \tilde{K}(C_g) \) mapping from and to generators of \( \tilde{K}(S^{2m}) \) and \( \tilde{K}(S^{2n}) \), respectively. Similarly, let \( a, b \in \tilde{H}^*(C_g; \mathbb{Q}) \) be elements mapping from and to generators of \( H^2m(S^{2m}; \mathbb{Z}) \) and \( H^2n(S^{2n}; \mathbb{Z}) \), respectively. We may assume \( \text{ch}(\alpha) = a \) and \( \text{ch}(\beta) = b + ra \) for some \( r \in \mathbb{Q} \). \( \beta \) and \( b \) are not uniquely determined, but if we vary them by integer multiples of \( \alpha \) and \( a \), we change \( r \) by an integers. So \( r \) is well-defined in \( \mathbb{Q}/\mathbb{Z} \).

We define \( e(g) = r \). One can check that it is a homomorphism.

Bounding the image of \( J \) below

The key to evaluating \( e(Jf) \) is the following lemma.

**Lemma:** \( C_{Jf} \) is the Thom space of the bundle \( E_f \to S^{2k} \) determined by the clutching function \( f : S^{2k-1} \to U(n) \). Under this identification, \( \beta \in \tilde{K}(C_{Jf}) \) corresponds to the Thom class of \( E_f \).

\( K(T(E_f)) \) is a free one-dimensional module over \( K(S^{2k}) \). It can be identified with a submodule of \( K(P(E_f)) \) generated by a specific relation corresponding to the Thom class. One may then apply the splitting principle and compute the value of the Chern character.

**Theorem:** (Atiyah?) Let \( E \) be any \( n \)-dimensional complex vector bundle with base \( B \). Let \( U \) denote the Thom class in \( H^*(T(E); \mathbb{Q}) \) which corresponds to \( 1 \in H^*(B; \mathbb{Q}) \) under the Thom isomorphism \( \Phi : H^*(B; \mathbb{Q}) \to \tilde{H}^*(T(E); \mathbb{Q}) \). Let \( bh_E \) denote the image of the characteristic class in \( H^*(BU(n); \mathbb{Q}) \) whose image in \( H^*(BU(1)^n; \mathbb{Q}) \) is

\[
\prod_{1 \leq r \leq n} \frac{e^{x_r} - 1}{x_r}
\]

Then

\[
\Phi^{-1}\text{ch}(U) = bh_E
\]

After some manipulation of power series, this implies

\[
e(Jf) = \alpha_k = \beta_k/k
\]
The $\alpha$-family

**Theorem:** Let $p$ be an odd prime, $m = p^f$, and $r = (p - 1)p^f$. Then there exists $\alpha \in \pi_{2r-1}^s$ such that

(i) $m\alpha = 0$

(ii) $e(\alpha) = -\frac{1}{m}$, and

(iii) The Toda bracket $\{m, \alpha, m\}$ is zero mod $m\pi_{2r}^s$.

For $q$ large, we have

$$\alpha : S^{2q+2r-2} \to S^{2q-1}$$

and the Toda bracket gives a map

$$S^{2q+2r-1} \to S^{2q-1}$$

Let $Y$ denote the cofiber of $m : S^{2q-1} \to S^{2q-1}$. Then since $m\{m, \alpha, m\} = 0$, the Toda bracket induces a map on the cofiber of $m$

$A : \Sigma^{2r}Y \to Y$

and we have a diagram

$$\begin{array}{ccc}
\Sigma^{2r}Y & \xrightarrow{A} & Y \\
\downarrow i & & \downarrow j \\
S^{2q+r-1} & \xrightarrow{\alpha} & S^{2q}
\end{array}$$

Adams defines the $d$ invariant of a map $f : X \to Y$ as $f^* \in \text{Hom}(K^*(Y), K^*(X))$. The $e$ invariant may be viewed as an element of a certain Ext group, and Adams shows that

$$d(jA) = -me(\alpha)$$

which in this case implies

$$d(jA) = 1$$

Thus, $A$ must be an isomorphism in $K$-theory.

Now, since $A$ is induces an isomorphism in $K$-theory, so does any composite

$$A \circ \Sigma^{2r}A \circ \Sigma^{4r}A \circ \cdots \circ \Sigma^{2r(s-1)}A : \Sigma^{2rs}Y \to Y$$

We may now construct a map $\alpha_s$ via the following diagram.

$$\begin{array}{ccc}
\Sigma^{2rs}Y & \longrightarrow & Y \\
\downarrow i & & \downarrow j \\
S^{2q+2rs-1} & \xrightarrow{\alpha_s} & S^{2q}
\end{array}$$

An argument like the one above shows that

$$e(\alpha_s) = -\frac{1}{m}$$

which shows that $\alpha_s$ is essential.