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Chromatic htpy thy
Introduction

π^s at primes 2, 3, 5 ← easier
↑ known range ~1000
known rang ~103

Patterns Im J

$J : \pi_* SO \to \pi_*^s$

$SO(n) \longrightarrow \Omega^n S^n$ $\quad A \mapsto A^+ : \mathbb{R}^{n+} \to \mathbb{R}^{n+}$

$SO \longrightarrow QS^0$ lifts to spectrum map

$\Sigma^{-1} bso \longrightarrow S^0$

$bso = K$ then kill π_0, and π_1

$BO \times \mathbb{Z} = K$ and then shift down

$\Sigma^{-1} bso$ is not a ring spectrum!

Adams $|\text{Im } J|_{4k-1} = \text{den}\left(\frac{B_k}{4k}\right)$

Global complicated

local easier

Fix p $(\pi_*^s)_p$

$$\begin{tikzcd}
 & \Sigma^{-1} bso \arrow[dl] \arrow[d, "J"] & \\
\Sigma^{-1} KO^\wedge_p \arrow[d] & S \arrow[dl, dashed] \arrow[d] & \\
J_p \arrow[r] & KO^\wedge_p \arrow[r, "\psi^\ell - 1"'] & KO^\wedge_p
\end{tikzcd}$$

htpy groups of J_p is basically the image of J

ℓ is a top gen of $\mathbb{Z}_p^\times / \{\pm 1\}$

Adams – Baird $\qquad S_{K/p} \simeq J_p$

2) Primes of homotopy theory

Alg $\qquad \operatorname{Spec} \mathbb{Z} \hookleftarrow \operatorname{Spec} \mathbb{Z}_{(p)} \hookleftarrow \operatorname{Spec} \mathbb{Q}$ $\qquad$ Spec $\mathbb{Z}$ is 1 dim'l

Topology "sphere is infinite dimensional"

$$S \longrightarrow S_p \xrightarrow[\substack{\infty\text{-many intermediate} \\ \text{localisations}}]{} S_{\mathbb{Q}}$$

$K(n) = n^{th}$ Morava K-thy

$BP_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n, \dots]$

$E(n) = BP/(v_{n+1}, v_{n+2}, \dots)[v_n^{-1}]$ Johnson-Wilson thy

$K(n) = E(n)/(p, v_1, v_2, \dots, v_{n-1})$

"chromatic tower"

$K(0) = E(0) = H\mathbb{Q}$

$S \longrightarrow S_{(p)} \longrightarrow \dots \longrightarrow S_{E(2)} \longrightarrow S_{E(1)} \longrightarrow S_{\mathbb{Q}}$

$\pi_* S_{(p)}$

$\text{filt}_n = \ker\left(\pi_* S_{(p)} \longrightarrow \pi_* S_{E(n-1)}\right)$

v_n-periodic layer

$v_1 = \text{im} J$ period $2(p-1)$

Q: (Arnov) Can you have an additive extension b/w periodic layers?

Periodicity in layers

Monochromatic layer

$$M_n S \to S_{E(n)} \to S_{E(n-1)}$$

Devinatz-Hopkins-Smith

$I = (i_0, \ldots, i_{n-1})$

$BP_* M_I = BP_* / (p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$

I cofinal, M_I exists

for $i_n \gg 0$, $\exists\ v_n^{i_n} : \Sigma^{2 i_n (p^n - 1)} M_I \to M_I$

iso on $E(n)$ homology

$\Rightarrow$ Non nilpotent self-map

Let $M_I^0 := \Sigma^{(-)} M_I$ $\qquad$ M_I^0 = Spanier-Whitehead dual of M_I

Fact: $M_n S = \varinjlim_I \left(M_I^0 \right)_{E(n)}$

$\bar{\bar{X}}$ $\quad$ $(M_I^0)_{E(n)}$ homology groups periodic

$$
\begin{array}{ccc}
S_{(p)} \ni X & & \\
\downarrow \quad \downarrow & & \\
S_{E(n)} & \xleftarrow{\ \bar{X}\ } & M_n S = \varinjlim_I (M_I^0)_{E(n)} \\
\downarrow & & \\
S_{E(n-1)} & &
\end{array}
$$

(4) Completions

M = f.g. $\mathbb{Z}$-mod

$$\begin{tikzcd} M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_p M_p^\wedge \arrow[d] \\ M_{\mathbb{Q}} \arrow[r] & \left(\prod_p M_p^\wedge\right)_{\mathbb{Q}} \end{tikzcd}$$

X spectrum

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_p X_p^\wedge \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & \left(\prod_p X_p^\wedge\right)_{\mathbb{Q}} \end{tikzcd}$$

htpy pull-back

$$\begin{tikzcd} X_{E(n)} \arrow[r] \arrow[d] & X_{K(n)} \arrow[d] \\ X_{E(n-1)} \arrow[r] & (X_{K(n)})_{E(n-1)} \end{tikzcd}$$

5) Moduli interpretation

ANSS $\quad \mathrm{Ext}_{MU_*MU}(MU_*, MU_*) \Rightarrow \pi_*^S$

$\quad\quad \cong$ // Quillen

$\quad H^*(\mathcal{M}_{FG})$

$\mathcal{M}_{FG}$ = moduli stack of 1-dim'l FG / $\mathrm{Spec}(\mathbb{Z})$

$$\mathcal{M}_{FG} \otimes \mathbb{F}_p \xhookleftarrow[\text{closed}]{} \mathcal{M}_{FG}^{\geq n}$$

$$\mathcal{M}_{FG}^{\leq n} = (\mathcal{M}_{FG})_{(p)} - \mathcal{M}_{FG}^{\geq n}$$

increasing sequence of opens

$$\mathcal{M}_{FG} \hookleftarrow (\mathcal{M}_{FG})_{(p)} \hookleftarrow \cdots \hookleftarrow \mathcal{M}_{FG}^{\leq 2} \hookleftarrow \mathcal{M}_{FG}^{\leq 1}$$
$$\hookleftarrow \mathcal{M}_{FG}^{\mathbb{Q}}$$

Adams-Novikov spectral sequence takes form

$$H^*(\mathcal{M}_{FG}^{\leq n}) \Rightarrow \pi_* S_{E(n)}$$

6) Formal moduli

$$S_{K(n)} \qquad M_n(S) \simeq M_n(S_{K(n)})$$

understanding these is in a sense the same

$$\mathcal{M}^{\leq n}_{FG} \xhookleftarrow[\text{closed}]{} \mathcal{M}^{=n}_{FG} = \mathcal{M}^{\leq n}_{FG} \cap \mathcal{M}^{\geq n}_{FG}$$

$$H^*\Big(\underbrace{\big(\mathcal{M}^{\leq n}_{FG}\big)^{\wedge}_{m=n}}_{\text{Formal nbhd of } \mathcal{M}^{=n} \text{ in } \mathcal{M}^{\leq n}}\Big) \Rightarrow \pi_* S_{K(n)}$$

So we are trying to understand deformations of ht n

$$\mathcal{M}^{=n} \otimes \overline{\mathbb{F}_p} = H_n^*$$ "Honda formal group of ht n"

$$\big\downarrow \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$$

$$\mathcal{M}^{=n}/\mathbb{F}_p$$

$\mathrm{Aut}(H_n) = \mathbb{S}_n =$ nth Morava Stabilizer group

<u>Lubin-Tate</u>: deformation of H_n are classified by ring

$$(E_n)_0 = W(\overline{\mathbb{F}_p})[[u_1, \ldots, u_{n-1}]]$$

$$\pi_* E_n = (E_n)_0 [u^{\pm 1}]$$

Morava change of rings $\Rightarrow$

$$H^*\left(\left(\mathcal{M}^{\leq n}\right)^{\wedge}_{\mathcal{M}^{=n}}\right) = H^*\left(\mathbb{G}_n, \pi_* E_n\right)$$

$$\mathbb{G}_n = \mathbb{S}_n \rtimes Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p)$$

Topological realization:

Hopkins-Miller thm

$$E_n \circlearrowleft \mathbb{G}_n$$

$$E_n^{h\mathbb{G}_n} \simeq S_{K(n)}$$

Then $H_c^*(\mathbb{G}_n, \pi_* E_n) \Rightarrow \pi_* S_{K(n)}$

e.g. $n=1$

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$$E_1^{hGal} = KU_p^\wedge \circlearrowleft \mathbb{Z}_p^\times = \mathbb{S}_1$$

actual int $\ell \in \mathbb{Z}_p^\times$

ℓ acts on $KU_p^\wedge$ by ℓth Adams operation

- G_n can be completely computed

Bad news

7) Bad primes

$n \rightsquigarrow \{\text{bad primes}\}$

$\mathbb{S}_n \hookleftarrow \mathbb{Z}/p^r$ $\quad r$ measure of badness

$r :=$ chromatic conductor

If n not divisible by $p-1$, n is good

$n = (p-1)p^{r-1}s \qquad (s,p)=1$

Badness: extra exotic / p-torsion / irregular periods

		Period
$n=1$	$p=2$	8

$2(p-1)$

$n=2$	$p=2$	$p=3$
	192	144

carries $\mathbb{Z}/p^n$

$H \leq G_n$ (finite)

E_n^{hH} captures badness — can be interesting to study

Ex $KO = E_1^{K\mathbb{Z}/2}$

$\pi_* KO = \mathbb{Z} \ \underbrace{\mathbb{Z}/2 \ \ \mathbb{Z}/2}_{\text{extra } \mathbb{Z}/2 \text{ in Hurewicz image}}$

$\pi_* tmf_{(2)}$ is 192 periodic which is irregular period at prime 2.