STABLE BUNDLES AND INTEGRABLE SYSTEMS

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§1. Introduction. The moduli spaces of stable vector bundles over a Riemann surface are algebraic varieties of a very special nature. They have been studied for the past twenty years from the point of view of algebraic geometry, number theory and the Yang-Mills equations. We adopt here another viewpoint, considering the symplectic geometry of their cotangent bundles. These turn out to be algebraically completely integrable Hamiltonian systems in a very natural way. For rank 2 bundles of odd degree this appeared as a byproduct of an investigation [5] into certain solutions of the self-dual Yang-Mills equations, a subject on which Yuri Manin has had a profound influence.

The cotangent bundle $T^*N$ of an $n$-dimensional complex manifold $N$ is a completely integrable Hamiltonian system if there exist $n$ functionally independent, Poisson-commuting holomorphic functions on $T^*N$. These functions for the case where $N$ is the moduli space of stable $G$-bundles on a compact Riemann surface $M$, and $G$ a complex semisimple Lie group are easy to describe. The tangent space of the moduli space at a point is identified with the sheaf cohomology group $H^1(M; \mathfrak{g})$ where $\mathfrak{g}$ is a holomorphic bundle of Lie algebras. By Serre duality the cotangent space is $H^0(M; \mathfrak{g} \otimes K)$. An invariant polynomial of degree $d$ on the Lie algebra then gives rise to a map from this cotangent space to the space $H^0(M; K^d)$ of differentials of degree $d$ on $M$. Taking a basis for the ring of invariant polynomials yields a map to the vector space $W = \bigoplus_{i=1}^{d} H^0(M; K^{d_i})$ where $d_i$ are the degrees of the basic invariant polynomials. Somewhat miraculously, the dimension of this vector space is always equal to the dimension of the moduli space $N$, thus providing the $n$ functions.

The Hamiltonian vector fields corresponding to these functions give $n$ commuting vector fields along the fibres of the map to $W$. The system is called algebraically completely integrable if the generic fibre is an open set in an abelian variety and the vector fields are linear. For the system above, at least for the case where $G$ is a classical group, this also turns out to be true, the abelian variety being either a Jacobian or a Prym variety of a curve covering $M$. The construction of this curve, and its corresponding Jacobian, parallels the solution of differential equations of “spinning top” type involving isospectral deformations of a matrix of polynomials in one variable. A point of the cotangent bundle of the moduli space consists of a stable vector bundle $V$ (with $G$-structure) and a holomorphic section $\Phi \in H^0(M; \mathfrak{g} \otimes K)$, which gives a holomorphic map $\Phi: V \to V \otimes K$. We form the curve of eigenvalues $S$ defined by the equation

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det(\lambda - \Phi) = 0 \text{ in the total space of the cotangent bundle of } M. \text{ Over } S \text{ there is a natural eigenspace line bundle } L \text{ which defines a point in the Jacobian of } S. \text{ Fixing the values of the invariant polynomials fixes the coefficients of this equation and hence also the curve. Thus the only further variation possible is to change the line bundle. For } G = \text{GL}(m, \mathbb{C}), \text{ the line bundle can take all possible values in the Jacobian. For } G = \text{Sp}(m, \mathbb{C}) \text{ or } \text{SO}(2m + 1, \mathbb{C}) \text{ the bundle is restricted to lie in the Prym variety of } S \text{ with respect to an involution with fixed points. For } G = \text{SO}(2m, \mathbb{C}), \text{ the curve } S \text{ is singular but the line bundle is defined on its desingularization, which has an involution without fixed points. It is the Prym variety of this curve which is a generic fibre in this case.}

Having established the existence of the integrable system we apply it in the realm of algebraic geometry to compute some sheaf cohomology groups of the moduli space of stable bundles of rank 2 and odd degree with fixed determinant. This consists of a generalization of the result of Narasimhan and Ramanan that \( H^0(N; T) = 0 \) and \( \dim H^1(N; T) = 3g - 3 \) to consideration of the corresponding groups for the \( k \)-th symmetric power bundle \( S^kT \). A consequence of this result is that the only globally defined holomorphic functions on the cotangent bundle of the moduli space, and polynomial in the fibres, are those generated by the functions defining the integrable system. Thus the global geometry of the moduli space determines the interesting symplectic geometry of its cotangent bundle.

The question which we have not treated here is to find explicitly the Hamiltonian differential equations which correspond to these systems. Since the moduli spaces are unirational (and conjectured to be rational) then ultimately these may be expressed as equations with rational coefficients. Finding some natural, concrete realization of the integrable systems which arise so naturally in this way may lead to an application in the other direction—from algebraic geometry to differential equations. This would be an agreeable outcome, and one consistent with Manin’s view of the unity of mathematics.

\section{Stable bundles}

2.1. Let \( M \) be a Riemann surface and \( V \) a \( C^\infty \) complex vector bundle of rank \( m \) over \( M \). By a holomorphic structure \( A \) on \( V \) we shall mean a differential operator

\[ d''_A : \Omega^0(M; V) \to \Omega^{0,1}(M; V) \]

such that

\[ d''_A(fs) = \overline{\partial f} \otimes s + fd''_A s \quad (2.2) \]

for \( s \in \Omega^0(M; V) \) and \( f \in C^\infty(M) \).

The local sections \( s \) satisfying \( d''_A s = 0 \) are defined to be holomorphic and with this definition one obtains holomorphic transition functions relating local holo-
morphic bases and consequently the usual definition of holomorphic bundle (see [2]).

2.3. Let \( \mathcal{G} \) denote the group of automorphisms of the vector bundle \( V \). The group acts on the space of holomorphic structures by

\[
d_A'' \rightarrow g^{-1}d_A''g.
\]

The space \( \mathcal{A} \) of all holomorphic structures on \( V \) is an infinite dimensional affine space since from 2.1

\[
d'' \rightarrow B \in \Omega^{0,1}(M; \text{End } V)
\]

and \( \mathcal{G} \) acts on \( \mathcal{A} \) via affine transformations.

2.4. If \( M \) is compact and of genus \( g > 1 \), then there is an open set \( \mathcal{A}^s \subset \mathcal{A} \) of holomorphic structures, the set of stable structures, which is preserved by \( \mathcal{G} \) and is such that the quotient \( \mathcal{A}^s/\mathcal{G} \) is a smooth complex manifold whose tangent space at \( A \) is isomorphic to the sheaf cohomology group \( H^1(M; \text{End } V) \).

The condition for stability is that for every proper subbundle \( U \),

\[
\frac{\deg(U)}{rkU} < \frac{\deg(V)}{rkV}.
\]

If equality occurs, the bundle is semistable.

2.6. The quotient space \( \mathcal{N} = \mathcal{A}^s/\mathcal{G} \) is the moduli space of stable vector bundles of rank \( m \) over \( M \). By the Riemann-Roch theorem (since stability implies \( H^0(M; \text{End } V) \equiv \mathbb{C} \)), the moduli space \( \mathcal{N} \) has dimension

\[
dim \mathcal{N} = m^2(g - 1) + 1.
\]

In the case where the rank \( m \) and Chern class \( c_1(V) \) are mutually prime, then \( \mathcal{N} \) is a compact projective variety.

2.8. The notion of stability may be extended from vector bundles to principal bundles ([2], [9]). In this case the space of holomorphic structures on a principal bundle \( P \) over \( M \) with structure group a complex semisimple group \( G \) is an infinite dimensional affine space with group of translations \( \Omega^{0,1}(M; \text{ad } P) \), where \( \text{ad } P \) is the vector bundle associated to the adjoint representation. For a Lie group \( G \), semistability of the principal bundle is equivalent to semistability of the holomorphic structure on the vector bundle \( \text{ad } P \).

In this more general situation there is a group \( \mathcal{G} \) of automorphisms of \( P \), with Lie algebra given by \( \Omega^0(M; \text{ad } P) \) and a subspace \( \mathcal{A}^s \) of stable holomorphic structures on \( P \) such that the quotient space \( \mathcal{A}^s/\mathcal{G} = \mathcal{N} \) is a smooth complex
quasi-projective variety with

$$\dim \mathcal{N} = \dim G (g - 1) \quad (2.9)$$

for a semisimple group $G$.

All these varieties have very special properties which have been investigated from many points of view ([2], [7], [8]). The point of view we adopt here is that of symplectic geometry in the holomorphic category. We shall consider the total space $T^*\mathcal{M}$ of the cotangent bundle of the moduli space of stable bundles as a complex symplectic manifold.

§3. Symplectic geometry. We recall some basic facts from symplectic geometry.

3.1. Let $N$ be any manifold and $T^*N$ its cotangent bundle with projection $\pi: T^*N \to N$. The tautological section of $\pi^*T^*N$ defines a 1-form $\theta$ on $T^*N$ such that $d\theta = \omega$ is a symplectic form on $T^*N$. If $(x_1, \ldots, x_n)$ are local coordinates on $N$ and cotangent vectors are parametrized by the coordinates $(y_1, \ldots, y_n) \to \sum_{i=1}^n y_i dx_i$, then the form $\theta$ is defined by

$$\theta = \sum y_i dx_i$$

and

$$\omega = \sum dy_i \wedge dx_i.$$

3.2. If $f$ is a smooth function on a symplectic manifold $M$, then it defines a vector field $X_f$:

$$df = i(X_f)\omega.$$ 

The Poisson bracket of two functions $f$ and $g$ is defined by the function

$$\{f, g\} = X_f \cdot g = -X_g \cdot f = -\{g, f\}.$$ 

Two functions are said to Poisson-commute if $\{f, g\} = 0$. A particular case is if $\omega = \sum dy_i \wedge dx_i$; and $f$ and $g$ are functions of $(y_1, \ldots, y_n)$ alone, for then

$$df = \sum \frac{\partial f}{\partial y_i} dy_i = i(X_f)(\sum dy_i \wedge dx_i)$$

and so

$$X_f = -\sum \frac{\partial f}{\partial y_i} \frac{\partial}{\partial x_i}.$$ 

Thus $\{f, g\} = X_f \cdot g = -\sum (\partial f/\partial y_i)(\partial g/\partial x_i) = 0$. 
3.3. If the vector fields $X_{f_1},\ldots, X_{f_n}$ form the basis of a Lie sub-algebra $\mathfrak{g}$ of the Lie algebra of vector fields on a symplectic manifold $M$, then the functions define a map to the dual of $\mathfrak{g}$, 

$$\mu: M \to \mathfrak{g}^*$$

given by

$$\mu(x) = \sum f_i(x) \xi_i$$

where $\{\xi_i\}$ is the dual basis in $\mathfrak{g}^*$.

When the vector fields integrate to give the action of a Lie group $G$ on $M$, and if $\mu$ is equivariant, then $\mu$ is called a moment map.

Suppose $G$ acts freely on the submanifold $\mu^{-1}(0)$. Then the symplectic form is degenerate along the orbits of $G$ in $\mu^{-1}(0)$ and invariant by $G$. It defines a symplectic form on the quotient $\mu^{-1}(0)/G$ which is called the Marsden-Weinstein quotient.

3.4. A special case of the Marsden-Weinstein quotient is when a Lie group $G$ acts freely on $N$, and hence on $T^*N$. A vector field $X$ on $N$ generated by this action defines a function from $T^*N$ by contraction in the fibres: if $X = \sum a_i(\partial/\partial x_i)$, $f = \sum a_i y_i$. The vector field $X_f$ on $T^*N$ is the natural extension of $X$ to the cotangent bundle.

In this case, the Marsden-Weinstein quotient is the cotangent bundle of the ordinary manifold quotient:

$$T^*(N/G) \equiv \mu^{-1}(0)/G.$$  

3.5. Let $M$ be a symplectic manifold with the action of a Lie group $G$ and moment map $\mu: M \to \mathfrak{g}^*$ defined by functions $f_1,\ldots, f_n$. If $g$ is a $G$-invariant function on $M$ then it defines by restriction a $G$-invariant function on $\mu^{-1}(0)$ and hence a function $\bar{g}$ on the Marsden-Weinstein quotient $\mu^{-1}(0)/G$.

Let $g, h$ be two such functions. Then $X_f g = 0$ since $g$ is $G$-invariant. But then

$$X_g f_i = \{ g, f_i \} = - X_{f_i} g = 0$$

so that $X_g$ is tangential to $f_i^{-1}(0)$ for all $i$ and thus to $\mu^{-1}(0)$.

If $\{ g, h \} = 0$, then $X_{\bar{g}} \cdot h = 0$ so $h$ is constant along the orbits of $X_{\bar{g}}$. But the projection of these orbits to $\mu^{-1}(0)/G$ are the orbits of $X_{\bar{g}}$. Thus $h$ is constant on the orbits of $X_{\bar{g}}$ and so $\{ \bar{g}, \bar{h} \} = 0$.

Thus if $g$ and $h$ Poisson-commute in $M$, so do $\bar{g}$ and $\bar{h}$ in $\mu^{-1}(0)/G$.

3.6. A symplectic manifold $M$ of dimension $2n$ is said to be a completely integrable Hamiltonian system if there exists functions $f_1,\ldots, f_n$ which Poisson-commute and for which $df_1 \wedge \cdots \wedge df_n$ is generically nonzero.
The map \( f: M \to \mathbb{C}^n \) defined by these functions—the moment map for an abelian Lie algebra—has the property that a generic fibre is an \( n \)-dimensional submanifold with \( n \) linearly independent commuting vector fields \( X_{f_1}, \ldots, X_{f_n} \). If this fibre is an open set in an abelian variety and the vector fields are linear, then we shall say that the system is \textit{algebraically completely integrable}.

We shall see that the cotangent bundle of the moduli space \( \mathcal{N} \) of stable bundles on a Riemann surface always has \( n = \dim \mathcal{N} \) natural Poisson-commuting functions and, at least for the classical groups, these form an algebraically completely integrable Hamiltonian system.

The most trivial example is the case of the complex group \( G = \mathbb{C}^* \). An equivalence class of holomorphic structures on a \( \mathbb{C}^* \)-bundle is defined by a point in the Picard group \( H^1(M; \mathcal{O}^*) \), or fixing the degree by a point in the Jacobian \( \text{Jac}(M) \). The tangent bundle of the Jacobian is trivial and isomorphic to \( \text{H}^0(M; K) \), and so by Serre duality the cotangent bundle is

\[
T^* \mathcal{N} \equiv \text{Jac}(M) \times H^0(M; K)
\]

where \( K \) is the canonical bundle.

The \( g \) functions obtained by projecting on the second factor Poisson-commute and every fibre of this map is a copy of the abelian variety which is the Jacobian. Every holomorphic vector field defined on a compact abelian variety is linear, so this is an algebraically completely integrable system, though not a very interesting one.

\section*{4. Invariant polynomials.} Let \( \mathcal{N} \) be the moduli space of stable holomorphic structures on a principal \( G \)-bundle \( P \). The tangent space at a point of \( \mathcal{N} \) may be represented by the vector space \( H^1(M; \text{ad } P) \) and hence by Serre duality the cotangent space is isomorphic to \( H^0(M; \text{ad } P \otimes K) \).

The vector bundle \( \text{ad } P \) is a bundle of Lie algebras each isomorphic to \( g \), and so if \( p \) is an invariant homogeneous polynomial on \( g \) of degree \( d \), it defines a map

\[
p: H^0(M; \text{ad } P \otimes K) \to H^0(M; K^d).
\]

Let \( p_1, \ldots, p_k \) be a \textit{basis} for the ring of invariant polynomials on the Lie algebra \( g \), then we obtain a map

\[
p: H^0(M; \text{ad } P \otimes K) \to \bigoplus_{i=1}^k H^0(M; K^{d_i}).
\]

\textbf{Proposition 4.1.} \textit{Let \( G \) be a semisimple complex Lie group and \( P \) a stable holomorphic principal \( G \)-bundle over a compact Riemann surface \( M \) of genus \( g > 1 \).}
Then

$$\dim H^0(M; \text{ad } P \otimes K) = \dim \bigoplus_{i=1}^{k} H^0(M; K^{d_i}).$$

**Proof.** If $P$ is stable, then $H^0(M; \text{ad } P) = 0$ (see [2]) and so by Riemann-Roch, using the Killing form to identify $\text{ad } P^* \equiv \text{ad } P$,

$$\dim H^0(M; \text{ad } P \otimes K) = \dim G(g - 1).$$  \hfill (4.2)

Since $G$ is semisimple there are no linear invariant polynomials, so each $d_i > 1$ and therefore applying Riemann-Roch to the right hand side

$$\dim \bigoplus_{i=1}^{k} H^0(M; K^{d_i}) = \sum_{i=1}^{k} (2d_i - 1)(g - 1).$$  \hfill (4.3)

But it is well known that

$$\dim G = \sum_{i=1}^{k} (2d_i - 1).$$

From a topological point of view, each invariant polynomial defines an invariant $(2d_i - 1)$-form on $G$ and the exterior algebra generated by these is the cohomology $H^*(G; \mathbb{C})$ [3]. More algebraically, Kostant's principal three-dimensional subgroup breaks up the Lie algebra $\mathfrak{g}$ into representation spaces of dimension $2d_i - 1$ [6]. From either source, the proposition is proved.

From 4.1, we obtain a map

$$p: T^*\mathcal{N} \to \bigoplus_{i=1}^{k} H^0(M; K^{d_i})$$  \hfill (4.4)

which provides $n$ holomorphic functions $f_i$ defined on the $2n$-dimensional symplectic manifold $T^*\mathcal{N}$.

**Proposition 4.5.** The $n$ functions $f_i$ on $T^*\mathcal{N}$ Poisson-commute.

**Proof.** From 3.4 we shall represent $T^*\mathcal{N}$ as a Marsden-Weinstein quotient:

$$T^*\mathcal{N} = T^*(\mathcal{A}^s/\mathcal{G}) = \mu^{-1}(0)/\mathcal{G}$$

where

$$\mu: T^*\mathcal{A}^s \to \mathfrak{g}^*$$

is the moment map for the action of $\mathcal{G}$ on the affine space $\mathcal{A}$. 

Now $T^*\mathfrak{g} \cong \mathfrak{g}^s \times \Omega^0(M; \text{ad } P \otimes K)$ with the canonical 1-form $\theta$ defined by

$$\theta(\dot{A}, \dot{\Phi}) = \int_M B(\dot{A} \wedge \dot{\Phi})$$

(4.6)

where $\dot{A} \in \Omega^{1,0}(M; \text{ad } P)$ is a tangent vector to $\mathfrak{g}^s$ at $\mathfrak{g}$ and $\dot{\Phi} \in \Omega^0(M; \text{ad } P \otimes K)$ a tangent vector to $\Omega^0(M; \text{ad } P \otimes K)$ at $\Phi$, and $B$ is the Killing form of the group.

Since $\mathcal{G}$ acts on $\mathfrak{g}^s$ by conjugation of the operator $d_A'$, the Lie algebra $\mathfrak{g} = \Omega^0(M; \text{ad } P)$ generates vector fields

$$\dot{A} = d_A' \psi \in \Omega^{0,1}(M; \text{ad } P)$$

for $\psi \in \Omega^0(M; \text{ad } P)$.

Consequently, from 3.4, $\mu(A, \Phi) = 0$ if and only if

$$0 = \int_M B(d_A' \psi \wedge \Phi) = \int_M B(\psi d_A' \Phi)$$

for all $\psi \in \Omega^0(M; \text{ad } P)$. Thus $\mu(A, \Phi) = 0$ if and only if $d_A' \Phi = 0$ i.e., $\Phi \in \Omega^0(M; \text{ad } P \otimes K)$ is holomorphic with respect to the holomorphic structure $d_A'$.

Taking a basis for the invariant polynomials on $\mathfrak{g}$ defines a map

$$q: \mathfrak{g}^s \times \Omega^0(M; \text{ad } P \otimes K) \to \bigoplus_{i=1}^k \Omega^0(M; K^{d_i})$$

simply by evaluating on the second factor. On the submanifold $\mu^{-1}(0)$ (i.e., the points $(A, \Phi)$ satisfying $d_A' \Phi = 0$) it takes values in the holomorphic sections of $K^{d_i}$ and, being invariant by $\mathcal{G}$, descends to $\mu^{-1}(0)/\mathcal{G} = T^*\mathcal{N}$ to define the map $p$ in (4.4). Now any two of these functions certainly Poisson-commute in $T^*\mathfrak{g}^s = \mathfrak{g}^s \times \Omega^0(M; \text{ad } P \otimes K)$ since they are functions of $\Omega^0(M; \text{ad } P \otimes K)$ alone as in 3.2. Therefore, by 3.5, they commute also on the quotient $T^*\mathcal{N}$, proving the proposition.

We see then that the 2n-dimensional symplectic manifold $T^*\mathcal{N}$ admits $n$ Poisson-commuting functions, where $\mathcal{N}$ is the moduli space of stable holomorphic structures on a principal $G$-bundle, for $G$ semisimple.

The same is true for $G = \text{GL}(m, \mathbb{C})$, i.e., the moduli space of stable bundles of rank $m$, the only difference being that the centre gives rise to a linear invariant, so that the dimension of the right hand side in 4.1 is $\sum_{i=1}^k (2d_i - 1)(g - 1) + g = (m^2 - 1)(g - 1) + g$, but by Riemann-Roch this is also the dimension of the left hand side. Restricting attention to vector bundles with fixed determinant bundle reduces this situation to that of the semisimple group $\text{SL}(m, \mathbb{C})$. 
§5. **Algebraic integrability.** We shall show now that the \( n \) Poisson-commuting functions defined above make \( T^* \mathcal{M} \) into an algebraically completely integrable Hamiltonian system for the classical groups \( G = \text{GL}(m, \mathbb{C}), \text{Sp}(m, \mathbb{C}), \text{SO}(2m, \mathbb{C}) \) and \( \text{SO}(2m + 1, \mathbb{C}) \). To do this we need to show that the generic fibre of the function \( p \) in (4.4) is \( n \)-dimensional (to obtain the functional independence of the functions \( f_j \)), and further that it is an open set in an abelian variety, with the vector fields \( X_j \) linear.

5.1. Let \( G = \text{GL}(m, \mathbb{C}) \). A point of \( T^* \mathcal{M} \) is given by a stable vector bundle \( V \) of rank \( m \) and a holomorphic section \( \Phi \in H^0(\mathbb{M}; \text{End} V \otimes K) \).

A basis for the invariant polynomials on \( \text{GL}(m, \mathbb{C}) \) is provided by the coefficients of the characteristic polynomial \( \det(x - A) \) of a matrix \( A \in \text{gl}(m, \mathbb{C}) \). There is one polynomial in each degree \( \leq m \), so we obtain the map

\[
p: T^* \mathcal{M} \to \bigoplus_{i=1}^{m} H^0(\mathbb{M}; K^i).
\]

To study the fibre of this map we take sections \( a_i \in H^0(\mathbb{M}; K^i) \) and consider the vector bundles \( V \) and sections \( \Phi \in H^0(\mathbb{M}; \text{End} V \otimes K) \) for which

\[
\det(x - \Phi) = x^m + a_1 x^{m-1} + \cdots + a_m.
\]  

(5.2)

Let \( X \) be the complex surface which is the total space of the cotangent bundle \( K_M \) of \( \mathbb{M} \). The pull-back of \( K_M \) to \( \mathbb{M} \) has a tautological section \( \lambda \). Consider the sections of the line bundle \( K^m_M \) on \( X \) of the form

\[
s = \lambda^m + a_1 \lambda^{m-1} + \cdots + a_m,
\]

for \( a_i \in H^0(\mathbb{M}; K^i) \).

As the \( a_i \) vary, the zero set of \( s \) forms a linear system of divisors on \( X \), and its compactification \( P(K^m_M \oplus 1) \). Any base point of this system must lie in the zero section of \( K_M \) since \( \lambda^m \) lies in the system. But then it is a base point for \( K^m_M \) on \( \mathbb{M} \) since \( a_m \) vanishes there for all \( a_m \in H^0(\mathbb{M}; K^m) \). But \( K^m \) is without base points for any Riemann surface, so the above system has no base points either. By Bertini’s theorem the generic divisor is a nonsingular curve \( S \).

The genus of \( S \) is given by the adjunction formula

\[
K_X \cdot S + S^2 = 2g(S) - 2.
\]

Now \( K_X = 0 \) (\( X \) is a symplectic manifold!) and \( S \) is in the linear system \( mK_M \). The zero section is in the linear system \( K_M \) and has self-intersection number \( K_M^2 = c_1(K_M) = 2g - 2 \), hence

\[
2g(S) - 2 = 2m^2(g - 1)
\]
and
\[ g(S) = m^2(g - 1) + 1. \]  
(5.3)

The curve \( S \) is an \( m \)-fold covering \( \pi: S \to M \) of \( M \). If \( \det(x - \Phi) = x^m + a_1 x^{m-1} + \cdots + a_m \), then pulling back the bundle \( V \) to \( S \), we have \( \Phi \in H^0(S; \text{End } V \otimes K_M) \) satisfying \( \det(\lambda - \Phi) = 0 \). Thus on \( S \), \( \Phi \) has the eigenvalue \( \lambda \in H^0(S; K_M) \). Generically the eigenvalues are distinct and so we obtain a one-dimensional eigenspace which defines a line bundle
\[ L \subset \ker(\lambda - \Phi) \subset \pi^* V \]
on \( S \) and hence a point of the Jacobian \( \text{Jac}(S) \), an abelian variety of dimension \( g(S) = m^2(g - 1) + 1 = \dim \mathcal{N} \) from (2.7).

Conversely, the line bundle \( L \) determines \( V \) and \( \Phi \) on \( M \). We consider the direct image sheaf \( \pi_* L \) which is the sheaf of sections of a vector bundle of rank \( m \) on \( M \). The fibre of the bundle \( \pi_* L \) at \( x \in M \) is by definition
\[ (\pi_* L)_x \equiv \mathcal{O}_S(L)/\mathcal{I}_{\pi^{-1}(x)} \]
where \( \mathcal{I}_{\pi^{-1}(x)} \) is the ideal sheaf of \( \pi^{-1}(x) \). If \( x \) is not a branch point then this is \( \bigoplus_{y \in \pi^{-1}(x)} L_y \). If \( x \) is a branch point, then the fibre is
\[ \bigoplus_{y \in \pi^{-1}(x)} J_{k(y)}(L)_y \]
a direct sum of jets of sections of \( L \) of degrees \( k(y) \) given by the degree of ramification of \( y \).

Each \( k \)-jet has a natural map to the 0-jet
\[ J_k(L)_y \to L_y \to 0 \]
and hence dually a map
\[ 0 \to L_y^* \to J_k(L)_y^* \]
There is thus a natural sequence for all \( x \)
\[ 0 \to \bigoplus_{y \in \pi^{-1}(x)} L_y^* \to (\pi_* L)_x^* \to \bigoplus_{y \in \pi^{-1}(x)} J_{k(y)}(L)_y^*/L_y^* \to 0. \]
This defines a sheaf map on \( M \)
\[ \mathcal{O}(\pi_* L)^* \to \mathcal{P} \to 0 \]  
(5.4)
where $\mathcal{S}$ is a sheaf supported at the branch points of $\pi: S \to M$ in $M$. The kernel of $s$ is a locally free sheaf $\mathcal{O}(W)$ of rank $m$.

If $L$ was obtained from a vector bundle $V$ and $\Phi \in H^0(M; \text{End} V \otimes K)$ as above, then the vector bundle $W$ is $V^*$, from the natural map $V^* \to \bigoplus_{y \in \pi^{-1}(x)} L_y^*$ induced by the inclusion $L \subset V$.

This way, we recover $V$ from the line bundle $L$. The operation of multiplication by $\lambda \in H^0(S; K_M)$ on $L$ yields a section of $\text{End} \pi_* L \otimes K_M$ which takes $W$ to $W \otimes K_M$ and so defines $\Phi \in H^0(M; \text{End} V \otimes K_M)$. We see therefore that the generic fibre of $p$ lies in an abelian variety of dimension $n = \text{dim } \mathcal{N}$. Since $p$ maps to an $n$-dimensional space this proves the functional independence of the Poisson-commuting functions $f_1, \ldots, f_n$ and shows that the fibre is an open set in the abelian variety. Not all vector bundles produced from line bundles on $S$ this way will be stable and it is the stable ones which determine the open set.

What remains is to show that the vector fields $X_{f_i}$ are linear on the Jacobian. This is equivalent to showing that they extend as holomorphic vector fields to the whole compact Jacobian.

Our construction of $V$ and $\Phi$ from a line bundle $L$ on $S$ was, however, natural in the sense that if a holomorphic isomorphism takes $V$ and $\Phi$ to $V'$ and $\Phi'$, then it takes the corresponding line bundle $L$ to $L'$. Thus the Jacobian is a space of equivalence classes of points in $\mathscr{A} \times \Omega^0(M; \text{End} V \otimes K)$ with $d\phi^* \Phi = 0$, under the action of the group of automorphisms $\mathscr{G}$. This space is strictly larger than $T^*\mathscr{A}^*$ which we used in §4, but by the Marsden-Weinstein construction still gives rise to a symplectic quotient. The cotangent bundle $T^*\mathscr{N}$ therefore lies as an open set in a larger symplectic manifold which contains the whole Jacobian and hence allows the vector fields $X_{f_i}$ to extend and therefore be linear. We shall consider this larger moduli space in more detail in the next section for the case $G = \text{GL}(2, \mathbb{C})$.

The Hamiltonian system is thus in this case algebraically completely integrable.

Finally note that the sheaf map (5.4) gives rise to an exact sequence of sheaves:

$$0 \to \mathcal{O}(V^*) \to \mathcal{O}(\pi_* L)^* \to \mathcal{S} \to 0$$

which determines the degree of the line bundle $L$.

From the Grothendieck-Riemann-Roch theorem,

$$\text{ch}(\pi_* L) \text{td}(M) = \pi_*(\text{ch}(L) \text{td}(S))$$

so

$$\text{ch}(\pi_* L) = (1 + (g - 1)[M]) \pi_*(1 + (\deg L + (1 - g(S)))[S])$$

and therefore

$$\deg \pi_* L = m(g - 1) + \deg L + (1 - g(S)).$$

(5.6)
On the other hand, the ramification divisor on $S$ is defined by the points whose tangent space projects to zero on $M$ and this is the divisor of a section of $K_S K^*_M$ which has degree

$$(2g(S) - 2) - m(2g - 2).$$

Thus from (5.5), since $S$ is supported on the branch points,

$$\deg V^* = \deg(\pi_* L)^* - \deg S$$

$$= m(g - 1) - (g(S) - 1) - \deg L$$

$$= -m(m - 1)(g - 1) - \deg L \text{ from (5.3).}$$

Hence

$$\deg L = -m(m - 1)(g - 1) - \deg V^*. \tag{5.9}$$

5.10. Let $G = \text{Sp}(m, \mathbb{C})$. A point of $T^* \mathcal{N}$ now consists of a stable vector bundle $V$ of rank $2m$ with a symplectic form $\langle , \rangle$, together with a holomorphic section $H(M; \text{End} V (\mathbb{C}) \otimes K)$ which satisfies $\langle \Phi v, w \rangle = -\langle v, \Phi w \rangle$.

Suppose $A \in \text{sp}(m, \mathbb{C})$ has distinct eigenvalues $\lambda_i$ with eigenvectors $v_i$ on $\mathbb{C}^{2m}$, then

$$\lambda_i \langle v_i, v_j \rangle = \langle Av_i, v_j \rangle = -\langle v_i, Av_j \rangle = -\lambda_j \langle v_i, v_j \rangle$$

so $\langle v_i, v_j \rangle = 0$ unless $\lambda_j = -\lambda_i$. Since $\langle v_i, v_i \rangle = 0$ it follows from the nondegeneracy of the symplectic inner product that if $\lambda_i$ is an eigenvalue so is $-\lambda_i$.

Thus the characteristic polynomial is of the form

$$\det(x - A) = x^{2m} + a_2 x^{2m-2} + \cdots + a_{2m}.$$ 

The polynomials $a_2, \ldots, a_{2m}$ in $A$ form a basis for the invariant polynomials on the Lie algebra $\text{sp}(m, \mathbb{C})$.

As in the previous section, we consider the linear system of divisors of the sections $\lambda^m + a_2 \lambda^{m-2} + \cdots + a_{2m}$ of $K^{2m}_M$ on the complex surface $X$ and by Bertini’s theorem see that the generic divisor is a nonsingular curve $S$ of genus $g(S) = 4m^2(g - 1) + 1$ from (5.3). We obtain again a line bundle

$$L \subset \ker(\lambda - \Phi) \subset \pi^* V$$

and thus a point on the Jacobian which determines $V$ and $\Phi$.

In this case, however, the curve $S$ with equation

$$\lambda^m + a_2 \lambda^{m-2} + \cdots + a_{2m} = 0$$
possesses the involution $\sigma(\lambda) = -\lambda$ with fixed points the zeros of $a_{2m} \in H^0(M; K^{2m})$ on the zero section $\lambda = 0$. The involution acts on the Jacobian of line bundles of degree zero. The subvariety of line bundles such that $\sigma^*L \cong L^*$ is an abelian variety—the Prym variety. Its dimension is $g(S) - g(S/\sigma)$.

Now $\sigma$ has $4m(g - 1)$ fixed points since they are the zeros of a section of $K^{2m}$ on $M$, so

$$2 - 2g(S) = 2(2 - 2g(S/\sigma)) - 4m(g - 1)$$

and thus

$$\dim \text{Prym}(S) = g(S) - g(S/\sigma)$$

$$= g(S) - \frac{1}{2} - \frac{1}{2}g(S) + m(g - 1)$$

$$= 2m^2(g - 1) + m(g - 1) \text{ from (5.3)}$$

$$= m(2m + 1)(g - 1). \quad (5.11)$$

In our situation the eigenspace bundle $L \subset \ker(\lambda - \Phi)$ actually defines a point in the Prym variety. This is because of the nondegenerate pairing above of eigenvectors corresponding to eigenvalues $\lambda$ and $-\lambda$. If $L$ is the eigenspace bundle corresponding to $\lambda$, then $\sigma^*L$ corresponds to $-\lambda$ and so the symplectic form defines a section of $L^* \otimes \sigma^*L^*$ which is nonvanishing if the eigenvalues are distinct i.e., away from the ramification locus of $S$.

Since $V$ is symplectic, $\deg V = 0$ and so from (5.7) and (5.8),

$$\deg L = -\frac{1}{2}\deg(K_SK_M^*) \cdot$$

Thus

$$\deg(L^* \otimes \sigma^*L^*) = \deg(K_SK_M^*)$$

and since $L^* \otimes \sigma^*L^*$ has a section which vanishes on the ramification locus, a divisor of $K_SK_M^*$, we have

$$\sigma^*L \cong L^* \otimes (K_SK_M^*)^{-1}. \quad (5.12)$$

Choosing a holomorphic square root $(K_SK_M^*)^{1/2}$ of $K_SK_M^*$ we obtain

$$L \cong U(K_SK_M^*)^{-1/2} \quad (5.13)$$

where $U$ is a line bundle such that $\sigma^*U \equiv U^*$, i.e., $U$ lies in the Prym variety.
Now from (5.11)

\[ \dim \text{Prym}(S) = m(2m + 1)(g - 1) \]
\[ = \dim \text{Sp}(m)(g - 1) \]
\[ = \dim \mathcal{N}. \]

Hence, the generic fibre of \( p \) is in this case an open set in an abelian variety—a Prym variety.

Note that in the converse construction of the vector bundle \( V \) from the line bundle \( L \), the involution \( \sigma \) produces a bilinear form on \( \pi_* L \) away from the branch locus by taking \( v \in H^0(p^{-1}(U); L) \) and setting

\[ \langle v, w \rangle_s = \sigma^* v \cdot w \in H^0(p^{-1}(U), K_S K_M^*) \]

from (5.12), where \( s \in H^0(S; K_S K_M^*) \) is the canonical section given by the derivative of \( \pi: S \to M \).

The form is skew because \( \sigma \) has real codimension 2 fixed points and is therefore of odd type [1] i.e., it lifts to an action on \( K_S^{1/2} \) of order 4. From (5.13) this means that \( \sigma^2 = -1 \) on \( L \).

5.14. Let \( G = \text{SO}(2m, \mathbb{C}) \). A point of \( T^* \mathcal{N} \) is now a stable vector bundle \( V \) of rank \( 2m \) with a nondegenerate symmetric bilinear form \( (\ , \ ) \), together with a holomorphic section \( \Phi \in H^0(M; \text{End} V \otimes K) \) which satisfies \( (\Phi v, w) = -(v, \Phi w) \).

Now suppose \( A \in \text{so}(2m, \mathbb{C}) \) has distinct eigenvalues \( \lambda_i \) with eigenvectors \( v_i \) on \( \mathbb{C}^{2m} \). Then

\[ \lambda_i(v_i, v_j) = (Av_i, v_j) = -(v_i, Av_j) = -\lambda_j(v_i, v_j) \]

so if \( \lambda_i \neq 0, (v_i, v_i) = 0 \) and, as in the previous case, if \( \lambda_i \) is an eigenvalue so is \(-\lambda_i\). Thus the characteristic polynomial is of the form

\[ \det(x - A) = x^{2m} + a_2 x^{2m-2} + \cdots + a_{2m}. \]

In this case, the polynomial \( a_{2m} = \det A \) of degree \( 2m \) is the square of a polynomial \( p_m — \text{the Pfaffian} — \) of degree \( m \). A basis for the invariant polynomials on \( \text{so}(2m, \mathbb{C}) \) is then given by the coefficients \( a_2, \ldots, a_{2m-2} \) and \( p_m \).

Now let \( p_m \in H^0(M; K^m) \) be a section with simple zeros and consider the linear system of divisors of the sections \( \lambda^2 + a_2 \lambda^2 + \cdots + a_{2m-2} + p_m \) for \( a_{2i} \in H^0(M; K^{2i}) \), \( 1 \leq i \leq m - 1 \). Each curve of this system has singularities where \( \lambda = 0 \) and \( p_m = 0 \). These are the base points of the system and so by Bertini’s theorem a generic divisor has these as its only singularities. Since \( p_m \) is a section of \( K^m \), there are \( \deg K^m = 2m(g - 1) \) singularities which are generically ordinary double points.
The curve $S$ in $X$ given by the generic divisor then has virtual genus given by the adjunction formula (5.3)

$$p = 4m^2(g - 1) + 1.$$ 

The genus of its nonsingular model $\hat{S}$ is thus

$$g(\hat{S}) = 4m^2(g - 1) + 1 - 2m(g - 1)$$

$$= 2m(2m - 1)(g - 1) + 1. \quad (5.15)$$

The involution $\sigma(\lambda) = -\lambda$ on $S$ has as fixed points the singularities of $S$ which are double points and so extends to an involution on $\hat{S}$ without fixed points.

We consider the Prym variety of $\hat{S}$. Its dimension is $g(\hat{S}) - g(\hat{S}/\sigma)$. Since $\sigma$ has no fixed points,

$$2 - 2g(\hat{S}) = 2(2 - 2g(\hat{S}/\sigma))$$

and thus

$$\dim \text{Prym}(\hat{S}) = g(\hat{S}) - g(\hat{S}/\sigma)$$

$$= g(\hat{S}) - \frac{1}{2} - \frac{1}{2}g(\hat{S})$$

$$= m(2m - 1)(g - 1). \quad (5.16)$$

Because $\hat{S}$ is smooth we obtain an eigenspace bundle $L \subset \ker(\lambda - \Phi)$ inside the vector bundle $V$ pulled back to $\hat{S}$, and as in the previous case, by applying the symmetric form, a section of $L^* \otimes \sigma^*L^*$ which is nonzero if the eigenvalues are distinct. It remains nonzero in fact when an eigenvalue tends to zero as may be seen by considering the eigenvectors of a $2 \times 2$ skew-symmetric matrix. Thus this section is nonvanishing away from the ramification points of the map $\pi: \hat{S} \to M$.

Arguing analogously to the symplectic case, we find

$$\sigma^*L \equiv L^* \otimes (K_{\hat{S}}K^*_M)^{-1}$$

and setting

$$L = U(K_{\hat{S}}K^*_M)^{-1/2}$$

we obtain a point of the Prym variety of $\hat{S}$.

Now from (5.16),

$$\dim \text{Prym}(\hat{S}) = m(2m - 1)(g - 1)$$

$$= \dim \text{SO}(2m)(g - 1)$$

$$= \dim \mathcal{N}.$$


We therefore obtain algebraic complete integrability in this case too. Since the involution $\sigma$ on $S$ now has no fixed points it is of even type and this is the reason that it provides a symmetric bilinear form on the corresponding vector bundle over $M$.

5.17. Let $G = \text{SO}(2m + 1, \mathbb{C})$. We now consider a stable vector bundle $V$ over $M$ of rank $(2m + 1)$ with a nondegenerate bilinear form $(\ , \ )$ and a holomorphic section $\Phi \in H^0(M; \text{End} V \otimes K)$ which satisfies $(\Phi v, w) = - (v, \Phi w)$.

By specializing from $\text{SO}(2m + 2)$, the characteristic polynomial of $A \in \text{so}(2m + 1)$ acting on $\mathbb{C}^{2m+1}$ is
\[
\det(x - A) = x(x^{2m} + a_2x^{2m-2} + \cdots + a_{2m})
\]
and the coefficients form a basis for the invariant polynomials on $\text{SO}(2m + 1)$.

As in the case of $\text{Sp}(m)$, we choose $a_{2i} \in H^0(M; K_M^{2i})$ for $1 \leq i \leq m$ such that the curve $S$ defined by
\[
\lambda^{2m} + a_2\lambda^{2m-2} + \cdots + a_{2m} = 0
\]
is nonsingular, and consider $V$ and $\Phi$ such that
\[
\det(\lambda - \Phi) = \lambda(\lambda^{2m} + a_2\lambda^{2m-2} + \cdots + a_{2m}).
\]
The vector bundle homomorphism $\Phi: V \rightarrow V \otimes K_M$ now always has an eigenvalue zero, so we automatically obtain a line bundle $V_0 \subset V$ defined by
\[
V_0 \subset \ker \Phi.
\]
The bundle $V$ on $M$ is therefore naturally an extension
\[
0 \rightarrow V_0 \rightarrow V \rightarrow V_1 \rightarrow 0. \quad (5.18)
\]
The zero eigenspace can be canonically identified for if we use the inner product on $V$ to convert the skew map $\Phi$ to a holomorphic section $\varphi \in H^0(M; \Lambda^2 V \otimes K_M)$, then
\[
\varphi \in H^0(M; \Lambda^{2m} V \otimes K_M^m) \cong H^0(M; V \otimes K_M^m)
\]
defines a homomorphism from $K_M^{-m}$ to $V$ whose image is the line bundle $V_0$. Thus $V_0 \cong K_M^{-m}$ and hence from (5.18) since $V = V^*$, we have
\[
\Lambda^{2m} V_1 \cong K_M^m. \quad (5.19)
\]
Since $V_0$ lies in the kernel of $\Phi$ we obtain an induced homomorphism $\Phi$:...
$V_1 \to V_1 \otimes K_M$ and we proceed as in 5.1 to obtain a line bundle $L$ on $S$ defining the eigenspace of $\Phi$.

The inner product $\langle \ , \rangle$ on $V$ defines a skew form on $V_1$ with values in $K_M$, by

$$\langle v, w \rangle = (\Phi v, w)$$  \hspace{1cm} (5.20)

for representative vectors $v$ and $w$ in $V$. Since $(\Phi v, w) = -(v, \Phi w)$ the form is clearly well defined on $V_1$ but is singular when $\Phi: V_1 \to V_1 \otimes K_M$ is singular which is when $a_{2m} = 0$. It nevertheless plays a role analogous to the symplectic form in 5.10 and just as in that case, if we use the involution $\sigma(\lambda) = -\lambda$ on $S$, we obtain a section of

$$L^* \otimes \sigma^* L^* \otimes K_M$$

from the pairing of the eigenvectors with eigenvalues $\pm \lambda$. Now from (5.9),

$$\deg L = -2m(2m - 1)(g - 1) - \deg V_1^*$$

$$= -2m(2m - 1)(g - 1) + m \cdot 2m \cdot 2(g - 1) \text{ from (5.19)}$$

$$= 2m(g - 1)$$

$$= \frac{1}{2} \deg K_M.$$ 

Thus $L^* \otimes \sigma^* L^* \otimes K_M$ is of degree zero with a nontrivial section and is hence a trivial bundle. Therefore

$$\sigma^* L \equiv L^* \otimes K_M.$$ 

Choosing a square root $K_M^{1/2}$ we obtain

$$L \equiv U(K_M^{1/2})$$  \hspace{1cm} (5.21)

where $U$ is a line bundle in the Prym variety of $S$. From (5.11)

$$\dim \text{Prym} S = m(2m + 1)(g - 1)$$

$$= \dim \text{SO}(2m + 1)(g - 1)$$

$$= \dim \mathcal{N}.$$ 

The methods of the previous cases allow us to recapture the rank $2m$ bundle $V_1$ from the line bundle $L$. It comes equipped with a skew form $\langle \ , \rangle$ with values in $K_M$ and a homomorphism $\Phi: V_1 \to V_1 \otimes K_M$ satisfying $\langle \Phi v, w \rangle = -\langle v, \Phi w \rangle$. We need to reconstruct the vector bundle $V$ of rank $(2m + 1)$. 
To do this we consider how $V_1$ arose and dualize the exact sequence (5.18) to obtain

$$0 \to V_1^* \to V \to V_0^* \to 0. \quad (5.22)$$

The composition $V_0 \to V \to V_0^*$ defines a section of $V_0^* \otimes V_0^* \cong K_M^{2m}$ which is easily seen to be the coefficient $a_{2m}$. On the zero set $D$ of this we have an inclusion $V_0 \subset V_1^*$.

Now consider the exact sequence of sheaves

$$0 \to \text{Hom}(V_0^*, V_1^*) \xrightarrow{a_{2m}} \text{Hom}(V_0, V_1^*) \to \text{Hom}(V_0, V_1^*)_D \to 0$$

and its exact cohomology sequence. The extension (5.21) is defined by the coboundary map

$$\delta: \text{H}^0(D; \text{Hom}(V_0, V_1^*)) \to \text{H}^1(M; \text{Hom}(V_0^*, V_1^*))$$

applied to the inclusion $V_0 \subset V_1^*$. An analogous construction occurs in the twistor approach to magnetic monopoles [4].

The advantages of this construction of $V$ is that it depends only on the bundle $V_1^*$ and the symmetric form induced on it from $V$ via the inclusion $V_1^* \subset V$. We consider a bundle $V_1^*$ of rank $2m$ with a symmetric form. The form degenerates where the corresponding homomorphism $g: V_1^* \to V$ is singular i.e., on a divisor $D$ of the line bundle $(\Lambda^{2m}V_1)$. Over $D$ there is a 1-dimensional subbundle $V_0 \subset V_1^*$, the kernel of $g$. The bundle $V_1^*/V_0$ has a nondegenerate symmetric form and so over $D$,

$$V_0 \cong \Lambda^{2m}V_1^*.$$

In the exact sequence of sheaves

$$0 \to \text{Hom}(\Lambda^{2m}V_1, V_1^*) \xrightarrow{\text{det } g} \text{Hom}(\Lambda^{2m}V_1^*, V_1^*) \to \text{Hom}(\Lambda^{2m}V_1^*, V_1^*)_D \to 0$$

the inclusion of $V_0 \cong \Lambda^{2m}V_1^*$ in $V_1^*$ over $D$ generates an extension

$$0 \to V_1^* \to V \to \Lambda^{2m}V_1 \to 0$$

by the coboundary map.

We use the above applied to the bundle $V_1$ arising from a point of the Prym variety of $S$. It has the skew form $\omega \in \text{H}^0(M; V_1^* \otimes V_1^* \otimes K_M)$ and the homomorphism $\Phi \in \text{H}^0(M; \text{End } V_1 \otimes K_M)$. We obtain a symmetric form by considering

$$\omega^{2m-1} \in \text{H}^0(M; \Lambda^{2m-1}V_1^* \otimes \Lambda^{2m-1}V_1^* \otimes K_M^{2m-1})$$

$$\cong \text{H}^0(M; V_1 \otimes V_1 ^* \otimes (\Lambda^{2m}V_1^*)^2 \otimes K_M^{2m-1})$$

$$= \text{H}^0(M; V_1 \otimes V_1 \otimes K_M^{-1})$$
and applying $\Phi$ to obtain

$$\Phi \omega^{2m-1} \in H^0(M; V_1 \otimes V_1).$$

Hence, a generic fibre of the map $p$ is an open set in the Prym variety of $S$ and the system is integrable.

**Remark.** Note the similarity between the cases of $G = \text{Sp}(m, \mathbb{C})$ and $\text{SO}(2m + 1, \mathbb{C})$. In both cases the projection $p$ maps to the same vector space of differentials on $M$ and the generic fibre is the same Prym variety: locally the two systems are equivalent. Globally, the difference in identifying the fibres as Prym varieties (compare (5.13) and (5.21)) will affect the structure of the moduli spaces. This may be a convenient context in which to bring in the duality theory of Lie groups by means of which $\text{Sp}(m, \mathbb{C})$ and $\text{SO}(2m + 1, \mathbb{C})$ are viewed as dual groups.

**§6. An application to moduli spaces.** We consider now an application of these symplectic methods—we use the integrable system to derive information about the sheaf cohomology groups of the moduli space $\mathcal{N}$ of stable vector bundles of rank 2 and odd degree over a Riemann surface of genus $g > 1$. This space is a compact smooth projective variety. If we fix the determinant line bundle $\Lambda^2 V$ of the stable vector bundle $V$, then it has complex dimension $3g - 3$.

In [7], Narasimhan and Ramanan proved that if $\mathcal{N}$ is the moduli space of stable bundles of rank $m$ and degree $k$, with $(m, k) = 1$ and with fixed determinant bundle, then the sheaf cohomology groups $H^i(\mathcal{N}; T)$ of the holomorphic tangent bundle $T$ vanish if $i \neq 1$ and $\dim H^1(\mathcal{N}; T) = 3g - 3$. In fact, the vanishing for $i > 1$ is a consequence of the vanishing theorem of Akizuki and Nakano, so the cases $i = 0$ and 1 are the only nontrivial ones. We shall generalize their result to deal with the $k$-th symmetric power bundle $S^k T$.

We restrict attention to the case of vector bundles of rank 2 because here, by analytical means, we have a precise description of the symplectic manifold alluded to in §5, in which $T^* \mathcal{N}$ lies as an open set [5]. We say that a pair $(A, \Phi) \in T^* \mathcal{A}$ consisting of a holomorphic structure $A$ and a section $\Phi \in \Omega^0(M; \text{ad } P \otimes K)$ satisfying $d_A^* \Phi = 0$ is a stable pair if for each $\Phi$-invariant holomorphic line bundle $L \subset V$, $\deg L < \frac{1}{2} \deg V$. Note from (2.5) that if the holomorphic structure is itself stable, then $(A, \Phi)$ is stable for all $\Phi$.

The quotient of the set of stable pairs with fixed determinant by the group $\mathcal{G}$ of automorphisms of $V$ of determinant one is, from [5], a complex symplectic manifold $\mathcal{M}$ of dimension $6g - 6$ with the following properties:

**Proposition 6.1.**

(i) $\mathcal{M}$ is a Kähler manifold.

(ii) The map $\det: \mathcal{M} \to H^0(M; K^2)$ defined by applying the invariant polynomial $\det$ on $\text{sl}(2, \mathbb{C})$ to $\Phi$ is proper and surjective.

(iii) The fibre of a point $q \in H^0(M; K^2)$ consisting of a section with simple zeros is isomorphic to the Prym variety of the double covering $S$ of $M$ defined by $\lambda^2 - q = 0$ in the total space $X = K_M$ of the cotangent bundle of $M$. 
The cotangent bundle $T^*\mathcal{N}$ of the moduli space of stable vector bundles lies naturally in $\mathcal{M}$ as the complement of an analytic set of codimension at least $g$.

Proof. For parts (i) to (iii), see [5], §8. In fact (iii) is essentially the description for $\text{SL}(2, \mathbb{C}) \cong \text{Sp}(1, \mathbb{C})$ in 5.10. The fact that $V$ is of odd degree only changes the isomorphism with the Prym variety.

For part (iv), if $V$ is unstable, then it has a canonical subbundle $L \subset V$ with $\deg L > \frac{1}{2}\deg V$. Thus $V$ is defined as an extension

$$0 \to L \to V \to L^* \otimes \Lambda^2 V \to 0$$

and hence by an element of $H^1(M; L^2 \otimes \Lambda^2 V^*)$.

Now the stability of the pair $(V, \Phi)$ implies from [5], (3.12), that

$$0 < \deg(L^2 \otimes \Lambda^2 V^*) < 2g - 2$$

(with strict inequality in this case since $\deg(V)$ is odd). This means that $\dim H^1(M; L^2 \otimes \Lambda^2 V^*) \leq g - 1$ with equality only if $L$ is special.

We need now another property of the symplectic manifold $\mathcal{M}$. Recall that a submanifold of a symplectic manifold is Lagrangian if its tangent spaces are isotropic of maximal dimension. The moduli space $\mathcal{M}$ is foliated by Lagrangian submanifolds (of dimension $(3g - 3)$) each of which is obtained by fixing the equivalence class of the complex structure and allowing $\Phi$ to vary (see [5], §8). On the open set $T^*\mathcal{N} \subset \mathcal{M}$ these manifolds are just the fibres of the cotangent bundle.

The moduli space $\mathcal{M}$ is stratified by the degree of the destabilizing subbundle $L$, so each stratum (for $\deg > 0$) is a bundle of $(3g - 3)$-dimensional spaces over equivalence classes of extensions of line bundles. Since $L$ is special there is at most a $(g - 1)$-dimensional choice for $L$ and since $\dim H^1(M; L^2 \otimes \Lambda^2 V^*)$ is less than $g$, at most a $(g - 2)$-dimensional family of extensions for a fixed $L$, giving the maximum dimension of stratum as $(3g - 3) + (g - 1) + (g - 2) = 5g - 6$, which is of codimension $g$ in $\mathcal{M}$.

Using these facts we prove the following:

Theorem 6.2. Let $\mathcal{N}$ be the moduli space of stable vector bundles with fixed determinant, rank 2 and odd degree over a compact Riemann surface of genus $g > 1$. If $T$ is the tangent bundle of $\mathcal{N}$ and $S^kT$ the $k$-th symmetric power of $T$, then

$$H^0(\mathcal{N}; S^kT) = 0 \text{ if } k \text{ is odd}$$

$$H^0(\mathcal{N}; S^2\ell T) \cong S^1H^1(M; T_M).$$

Proof. Let $s$ be a holomorphic section of $S^kT$ on $\mathcal{N}$. By contraction it defines a holomorphic function $\tilde{s}$ on the cotangent bundle $T^*\mathcal{N}$, which is a polynomial of degree $k$ on each fibre.
Since $\mathcal{M}$, the moduli space of stable pairs, is obtained by adjoining an analytic set of codimension $g \geq 2$ to $T^*\mathcal{M}$, then $\hat{s}$ extends by Hartog's theorem to a holomorphic function on $\mathcal{M}$.

Consider the map $\det: \mathcal{M} \to H^0(M; K^2)$. The generic fibre is compact and connected—a Prym variety—and hence, since $\det$ is proper and $H^0(M; K^2)$ smooth, every fibre is compact and connected. Thus the function $\hat{s}$ is constant on each fibre and so

$$\hat{s} = f \circ \det$$

for some holomorphic function $f: H^0(M; K^2) \to \mathbb{C}$.

Now the map $\Phi \to \lambda \Phi$ induces a $\mathbb{C}^*$ action on $\mathcal{M}$ and since $\lambda$ was homogeneous of degree $k$, we have

$$\hat{s}(\lambda \cdot m) = \lambda^k \hat{s}(m) \quad \text{for } m \in \mathcal{M}.$$ 

But $\det(\lambda \cdot m) = \lambda^2 \det(m)$, and so $f$ satisfies

$$f(\lambda^2 x) = \lambda^k f(x). \quad (6.3)$$

Now a holomorphic function on a vector space $W$ which is homogeneous of degree $\ell$ defines a holomorphic section of $\mathcal{O}(\ell)$ on $P(W)$ and hence a polynomial of degree $\ell$. Since $\mathcal{O}(1)$ generates $H^1(P(W), \mathcal{O}^*)$ there are no homogeneous functions of noninteger degree, so if $k$ is odd, $f \equiv 0$. If $k = 2\ell$, then $f$ is a polynomial of degree $\ell$ on $W = H^0(M; K^2)$ and hence an element of the vector space $S^\ell W^*$. Thus, since $H^0(M; K^2)^* \cong H^1(M; K^{-1})$ by Serre duality, we have $\hat{s} = 0$ if $M$ is odd and $\hat{s} = f \circ \det$ for $f \in S^\ell H^1(M; K^{-1})$ for $m = 2\ell$ and hence the theorem.

**Remark 6.4.** (1) Theorem 6.2 gives $H^0(\mathcal{M}; S^2T) \cong H^1(M; K^{-1})$. This shows that the $3g - 3$ functions which make $T^*\mathcal{M}$ into an algebraically completely integrable Hamiltonian system are globally determined by $\mathcal{M}$ itself: they are the only holomorphic functions on $T^*\mathcal{M}$ which are homogeneous of degree 2 in the fibres. Thus if $\mathcal{N}$ is given in some way other than as the moduli space of stable vector bundles, it is still possible to describe $T^*\mathcal{N}$ as an integrable system simply by finding the global sections of $S^2T$. An example would be the case of $g = 2$ where $\mathcal{N}$ is the intersection of two quadrics in $P^5$ [8].

(2) If we consider the ring of all holomorphic functions on $T^*\mathcal{N}$ which are polynomial in the fibres, then Theorem 6.2 shows that this ring is generated by the $3g - 3$ functions of degree 2. Since they Poisson-commute by 4.5, the whole Poisson algebra of polynomial functions is commutative.

In a similar manner we prove now the following:

**Theorem 6.5.** Let $\mathcal{N}$ be as in 6.2, and suppose that $g > 2$. Then

(i) $H^1(\mathcal{N}; S^kT) = 0$ if $k$ is even,
(ii) $H^1(\mathcal{N}; T) \equiv H^1(M; T_M),$
(iii) $H^1(\mathcal{N}; S^{2\ell+1}T) \equiv H^1(\mathcal{N}; T) \otimes H^0(\mathcal{N}; S^{2\ell}T).$
Proof. We begin as in Theorem 6.2. Each cohomology class \( \alpha \in H^1(\mathcal{N}; S^kT) \) defines by contraction a class \( \hat{\alpha} \in H^1(T^*\mathcal{N}; \mathcal{O}) \) which is homogeneous of degree \( k \) under the \( \mathbb{C}^* \)-action.

We may use the cohomological version of Hartog's theorem and extend this over an analytic set of codimension \( \geq 3 \), provided by Proposition 6.1 to a class \( \alpha \in H^1(\mathcal{M}; \mathcal{O}) \) [10]. This class is determined by its restriction to the complement of an analytic set of codimension \( \geq 2 \) and so is determined by its restriction to \( T^*\mathcal{N} \) or even to the complement of the zero section in \( T^*\mathcal{N} \). Thus any \( \beta \in H^1(\mathcal{M}; \mathcal{O}) \), homogeneous of degree \( k \) is determined by its restriction to the complement of the zero section in \( T^*\mathcal{N} \) and hence by the class in \( H^1(P(T^*\mathcal{N}); \mathcal{O}(k)) \) which it defines, where \( \mathcal{O}(1) \) is the tautological hyperplane bundle along the fibres.

Since \( H^i(\mathcal{N}; \mathcal{O}(k)) = 0 \) for \( i > 0 \), the Leray spectral sequence identifies this group with \( H^1(\mathcal{N}; S^kT) \). We may therefore simply consider classes \( \beta \in H^1(\mathcal{M}; \mathcal{O}) \) homogeneous of degree \( k \).

We consider the map \( \det: \mathcal{M} \to H^0(M; K^{-2}) = V \) which defines the integrable system and the first direct image sheaf \( R^1\det_\ast \mathcal{O} \). Since \( \det \) maps to a vector space \( V \), all the higher cohomology groups of direct image sheaves vanish, and the Leray spectral sequence gives

\[
H^1(\mathcal{M}; \mathcal{O}) \cong H^0(V; R^1\det_\ast \mathcal{O}). \tag{6.6}
\]

We shall prove that, outside an analytic set of codimension 2 in \( V \), the sheaf \( R^1\det_\ast \mathcal{O} \) is isomorphic to the sheaf of holomorphic functions with values in \( H^1(M; K^{-1}) \). Since elements of \( H^1(\mathcal{M}; \mathcal{O}) \) are determined by their restriction to complements of codimension 2 sets this will, using Hartog's theorem, provide from (6.6) an isomorphism

\[
H^1(\mathcal{M}; \mathcal{O}) \cong H^0(V; \mathcal{O}) \otimes H^1(M; K^{-1}). \tag{6.7}
\]

To obtain this isomorphism we use the Kähler form to define a class \( w \) in \( H^1(\mathcal{M}; T^*) \). Contraction with a holomorphic vector field gives a homomorphism from \( H^0(\mathcal{M}; T) \) to \( H^1(\mathcal{M}; \mathcal{O}) \). We apply this to the vector space of Hamiltonian vector fields arising from linear functions on \( V \): the basic fields of the integrable system. This defines a linear map from \( V^* \cong H^1(M; K^{-1}) \) to \( H^1(\mathcal{M}; \mathcal{O}) \).

Let \( q \in H^0(M; K^2) \) have simple zeros, then the fibre \( \mathcal{M}_q \) of \( \det \) over \( q \) is isomorphic to a Prym variety, and the vector fields generated by \( V^* \) are tangent to \( \mathcal{M}_q \) and span the space \( H^0(\mathcal{M}_q; T) \) of all holomorphic vector fields on this abelian variety. Since the Kähler form restricts on \( \mathcal{M}_q \) to a Kähler form it defines an isomorphism:

\[
H^0(\mathcal{M}_q; T) \xrightarrow{w} H^1(\mathcal{M}_q; \mathcal{O}) \cong \mathbb{C}^{3g-3}. \tag{6.8}
\]
This gives an isomorphism of $R^i\det_*\mathcal{O}$ with the functions with values in $V^*$ on the open set of $q \in H^0(M; K^2)$ with simple zeros. This is the complement of a codimension 1 analytic set.

We need to consider more degenerate quadratic differentials. If $g > 3$, then the $q \in H^0(M; K^2)$ which have at most one double zero is the complement of a codimension 2 set, so we must consider the singular fibre $\mathcal{M}_q$ for $q$ with a double zero at $x \in M$ and simple zeros elsewhere. We consider therefore a stable pair $(V, \Phi)$ on $M$ such that $\det \Phi$ has a double zero at $x$.

The curve $S \subset X = K_M$ defined as in §5 now has an ordinary double point over $x$, but we obtain an eigenspace line bundle $L$ over its desingularization $\hat{S}$, which has genus $(4g - 3) - 1 = (4g - 4)$. By arguments similar to those in (5.10), $L \cong U \otimes L_0$ for some fixed $L_0$ where $U$ lies in the $(3g - 4)$-dimensional Prym variety of $\hat{S}$. If $\Phi(x)$ is nonzero, it is nilpotent and has a unique eigenspace in $V_x$. This provides an isomorphism between $L_x$ and $L_{ax}$ on $\hat{S}$ (and hence in fact a point on the Prym variety of the singular curve $S$).

Since $\sigma^*L \cong \sigma^*U \otimes L_0 \cong U^{-1} \otimes L_0 \cong L^* \otimes L_0^2$, an isomorphism between $L_x$ and $L_{ax}$ is a nonzero section of $(L_x^2 \otimes L_0^{-2})_x$. But the Poincaré bundle over $M \times \text{Jac}(M)$ is trivial on $\{x\} \times \text{Jac}(M)$, so if $\Phi(x)$ is nonzero, it corresponds to an open set of $\mathcal{M}_q$ isomorphic to $\mathbb{C}^* \times \text{Prym}(\hat{S})$.

The remaining case is if $\Phi(x)$ vanishes. This must occur since the fibre of $\det$ is compact. In this case we can put $\Phi = s\Psi$ where $s$ is a section of the line bundle $\mathcal{O}(x)$ vanishing at $x$ and $\Psi \in H^0(M; \text{ad } P \otimes K(-x))$. A repetition of 5.10 shows that such a stable pair is determined by a point in the Prym variety $\text{Prym}(\hat{S})$, since the nonsingular curve $\hat{S}$ is the double cover of $M$ branched over the simple zeros of $q = \det \Phi$. Thus each $\mathbb{C}^*$ is compactified by a single point and we find

$$\mathcal{M}_q \cong E \times \text{Prym}(\hat{S})$$

where $E$ is a rational curve with an ordinary double point: a degenerate elliptic curve.

Now $H^1(\mathcal{M}_q; \mathcal{O}) \cong H^1(E; \mathcal{O}) \oplus H^1(\text{Prym}(\hat{S}); \mathcal{O}) \cong \mathbb{C}^{3g-3}$ so $R^1\det_*\mathcal{O}$ is locally free on the open set of $q$ with at most one double zero. Also, the product with $w \in H^1(\mathcal{M}; T^*)$ maps the vector fields tangent to $\text{Prym}(\hat{S})$ isomorphically to $H^1(\text{Prym}(\hat{S}); \mathcal{O})$ as before, and by direct calculation certainly maps the unique holomorphic vector field on the singular curve $E$ to a nonzero element in $H^1(E; \mathcal{O}) \cong \mathbb{C}$.

Thus the isomorphism of $R^1\det_*\mathcal{O}$ with the sheaf of functions extends to the complement of a codimension 2 set, and we obtain the isomorphism (6.7).

It remains to detect the classes in $H^1(\mathcal{M}; \mathcal{O})$ which are homogeneous of degree $k$. Now the canonical symplectic form $\omega$ on $T^*\mathcal{N}$ is of degree 1 (see (5.11)). Since a function $f$ generates a Hamiltonian vector field $X$ by $i(X)\omega = df$, then the functions $f \in V^* \cong H^1(M; K^{-1})$ which are homogeneous of degree 2 generate vector fields homogeneous of degree 1. It follows that the subspace of
$H^1(\mathcal{M}; \mathcal{O})$ of degree $k$ is isomorphic in (6.7) with the tensor product of $H^1(M; K^{-1})$ with holomorphic functions on $V$ of degree $\frac{1}{2}(k - 1)$. The theorem then follows, using for part (iii) the result of Theorem 6.2.

**Remarks.** (1) The natural isomorphism $H^1(\mathcal{N}'; T) \cong H^1(M; T_M)$ is almost certainly the Kodaira-Spencer map which associates to a deformation of the complex structure of the Riemann surface $M$, a deformation of the moduli space $\mathcal{N}$ (see [7]).

(2) The third part of the theorem says that $\bigoplus_k H^1(\mathcal{N}'; S^k T)$ is a free module over $\bigoplus_k H^0(\mathcal{N}'; S^k T)$ generated by $H^1(\mathcal{N}'; T)$.

**REFERENCES**


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