



FIGURE 1. Picture of where this talk fits in the Chern-Simons story.

1. CHERN-SIMONS THEORY AND THE CATEGORIFIED GROUP RING,
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The talk is about the paper by FHT and Lurie. Extended notes of the talk are also available on my webpage.

So as we've been hearing today, we would like to extend Chern-Simons theory to the point. For all this there will be a parameter in the theory,

$$\tau \in H^4(BG; \mathbb{Z}).$$

There appear to be two problems to take care of to implement this:

- (1) Lurie: Whatever we assign to a point determines the whole theory, but we already know Chern-Simons theory at higher levels, so we must be very careful with our choice.
- (2) The anomaly: Chern-Simons theory is *not* a proper 3d TQFT.

The above mentioned paper says we should look at:

- (1) The case of a finite group (also called Dijkgraaf-Witten theory)
- (2) Look at classical CS-theory and quantize
- (3) We should decategorify and look for similar cases of 1d- and 2d TQFTs

There is a picture to keep in mind: have a horizontal axis labeling the dimension of the theory, a vertical axis of going from classical to quantum, and a third axis from finite to compact groups. We want to end up at the point the furthest from the origin. In this talk we start with a 2-dimensional quantum theory on finite groups, categorify this to a 3-dimensional theory on finite groups, and then hint about how to go to general compact groups.

The method we use is geometric realizations of twisting parameter, τ , by geometrically realizing $H^{n+1}(BG; \mathbb{Z})$. We have a table:

n	Group Theoretical Realization	Geometrical Realization
1	character of G	multiplicative map $G \rightarrow S^1$
2	central extension of G by S^1	multiplicative S^1 -bundle over G , (K, θ)
3	<i>String</i> extension of G by BS^1	multiplicative S^1 -gerbe over G

So let's think about how we construct these: a line bundle is a cover with transition functions, and a gerbe is a cover with "transition bundles" on 2-fold intersections, and we can iterate this idea as we wish.

Definition 1.1. A multiplicative gerbe is

- (1) a gerbe \mathcal{G} over a group G (like a twist of $K(G)$, if you like)
- (2) a 1-isomorphism over $G \times G$ "fiberwise:"

$$K_{x,y} : \mathcal{G}_x \otimes \mathcal{G}_y \rightarrow \mathcal{G}_{xy}$$

- (3) a 2-isomorphism over $G \times G \times G$:

$$\begin{array}{ccc} \mathcal{G}_x \otimes \mathcal{G}_y \otimes \mathcal{G}_z & \xrightarrow{K_{x,y} \otimes id} & \mathcal{G}_{xy} \otimes \mathcal{G}_z \\ id \otimes K_{y,z} \downarrow & \swarrow \theta_{x,y,z} & \downarrow K_{xy,z} \\ \mathcal{G}_x \otimes \mathcal{G}_{yz} & \xrightarrow{K} & \mathcal{G}_{xyz} \end{array}$$

- (4) and a pentagon axiom for θ over $G \times G \times G \times G$.

As a remark, all the models we've been taking for twists in K -theory are gerbes. Or if you like, gerbes are geometric realizations of $H^3(G; \mathbb{Z})$.

Now, the underlying gerbe may be trivial even though the multiplicative gerbe is not. So what is a multiplicative structure on a trivial gerbe? A multiplicative structure on a trivial gerbe is a pair (K, θ) with K an S^1 -bundle over $G \times G$ and θ an isomorphism:

$$\theta_{x,y,z} : K_{xy,z} \otimes K_{x,y} \rightarrow K_{x,yz} \otimes K_{y,z}$$

satisfying the pentagon. Note that composition of gerbe morphisms becomes tensor product of line bundles.

The G -equivariant structure on the trivial multiplicative gerbe (K, θ) is an S^1 -line bundle L over $G \times G$ is given by

$$L_{g,x} = K_{g \times g^{-1}, g}^* \otimes K_{g,x}$$

where we write (g, x) to emphasize that although both are group elements, g is acting on x .

1.1. 2-Dimensional TQFTs for Finite Groups.

Theorem 1.2. *There is an equivalence*

$$\begin{aligned} & \{0 - 1 - 2 \text{ TQFTs for oriented manifolds with target } Alg\} \\ \stackrel{\text{Lurie}}{\cong} & \{\text{Dualizable objects in } Alg \text{ homotopy fixed under } SO(2)\} \\ \cong & \{\text{semi - simple Frobenius algebra}\} \end{aligned}$$

Above Alg is the 2-category of algebras, bimodules, and bimodule maps. We're eventually going to categorify this picture to get to the 3-dimensional theory.

Now choose a finite group G and form the group

$$A = \mathbb{C}^\tau[G] = \Gamma(G; K_{\mathbb{C}})$$

where for the K we got from our gerbe (and hence our twist τ) we define

$$K_{\mathbb{C}} := K \times_{S^1} \times \mathbb{C}.$$

We get a ring multiplication:

$$(\gamma_1 \star \gamma_2)(g) = \sum_{h \in G} \theta(\gamma_1(gh^{-1}) \otimes \gamma_2(h)).$$

This is the “twisted group ring.” We get a trace,

$$\text{tr}(\gamma) = \frac{1}{|G|} \gamma(e) \in K_{\mathbb{C}|_e} \stackrel{\theta}{\cong} \mathbb{C}.$$

From the theorem we get an associated 2-d TQFT: to a point we assign $A = \mathbb{C}^\tau[G]$; the circle gets sent to $A \otimes_{A \otimes A^{op}} A \cong Z(A)$ where we use the trace to identify A with its dual; and Σ is sent to

$$\sum_{[P]} \frac{1}{|Aut(P)|} e^{2\pi i \langle \Sigma, \xi_P \tau \rangle}$$

where $\xi_P : \Sigma \rightarrow BG$ classifies the principle bundle P and we regard $\tau \in H^2(BG; S^1)$.

1.2. Categorification: From 2-d to 3-d. The theme here is to replace elements of \mathbb{C} to objects of $Vect$. So we take as our target category the category of 2-algebras (here we’re using tensor categories as a model for 2-algebras). Now we need to find a (fully) dualizable object. We want to go from the twisted group ring to the category Constantin talked about, $Vect^\tau[G]$. Now,

$$Vect^\tau[G] := VBun(G)$$

where we take $VBun(G)$ as a category. This is very bad definition when G is not finite, and we’ll need some sheafy thing. For now let’s not worry so much. We need a monoidal structure on $Vect^\tau[G]$:

$$(W \star W')_g := \bigoplus_{hh'=g} K_{\mathbb{C}|_{h,h'}}^\tau \otimes W_h \otimes W'.$$

You can see better motivation for all this coming from the quantization of a classical theory (via the picture mentioned in the introduction).

Now we want to prove that $Vect^\tau[G]$ is dualizable and does in fact lead to Chern-Simons theory.

Theorem 1.3.

- (1) *The 3d TQFT defined by $Vect^\tau[G]$ assigns to the circle the Drinfeld center*

$$S^1 \mapsto Z(Vect^\tau[G]).$$

- (2) *The Drinfeld center Z consists of vector bundles W over G together with isomorphisms*

$$L_{x,y} \otimes W_x \rightarrow W_{yxy^{-1}}.$$

Corollary 1.4. $K(Z(Vect^\tau[G])) \cong {}^\tau K_G^0(G)$

Corollary 1.5. “ $Z(Vect^\tau[G]) \cong PER^\tau(LG)$ ”

We're being a bit imprecise here with equivalences of categories versus isos on K -theory. Need to do some sum over conjugacy classes and then prove product structures are the same, or something. Also there are issues with G being disconnected in discussing $PER^\tau(LG)$.

Corollary 1.6. *The 3d TQFT we get is Chern-Simons theory.*

Let's try to sketch the proof of the above theorem. The 1st part Chris has already discussed, so let's look at the 2nd part. A Drinfeld center is the natural generalization of the center of an algebra to the center of a monoidal category. Then for $(W, \epsilon) \in Z(\text{Vect}^\tau[G])$:

$$\epsilon_x : K_{xy^{-1}, y} \otimes W_{xy^{-1}} \rightarrow K_{y, y^{-1}x} \otimes W_{y^{-1}x}$$

and for $W' = \mathbb{C}_y$,

$$\epsilon_{xy} : K_{yxy^{-1}, y} \otimes W_{yxy^{-1}} \rightarrow K_{y, x} \otimes W.$$

1.3. A Quick Tour of the 3d Theory in the Case of a Torus. So now, let's consider what happens when $G = T = S^1 \times S^1$. We want vector bundles supported over only finitely many points on the torus, and these are skyscraper sheaves. So define the category $Sky^\tau[G]$ as the category of skyscraper sheaves with values in finite dimensional vector spaces supported over finitely many points.

For example, for $y \in T$, $\mathbb{C}_y \in Sky^\tau[T]$. Furthermore, these guys generate the category from the monoidal structure, which we define to be:

$$\mathbb{C}_x \star \mathbb{C}_y := K_{x, y} \otimes \mathbb{C}_{xy}.$$

Now we run into some problems:

- (1) $Sky^\tau[T]$ is not dualizable in $2 - Alg$, so does not define a field theory, This leads us to an anomaly.
- (2) $Z(Sky^\tau[T]) \cong Sky^\tau[\mathfrak{t}] \otimes Sky^\tau[F]$ where F is a character lattice of the torus. But, $Sky^\tau[F]$ is the modular tensor category we want to assign to a circle, so $Sky^\tau[T]$ is somehow too big.

Let's explain this anomaly. The solution we take is to view the above "theory" as a morphism from the trivial theory to some anomaly theory:

$$1 \xrightarrow{CS^\tau} \mathcal{A}_\tau.$$

Now, a transformation between 4-functors is a 3-functor, so the dimensions are correct. Thus, we need to describe the 4-dimensional TQFT \mathcal{A}_τ . It assigns to the point our "unwanted factor:"

$$pt \xrightarrow{A_\tau} Sky^\tau[\mathfrak{t}],$$

and so we would need to show that $Sky^\tau[\mathfrak{t}]$ is dualizable in 3-algebras, fixed under $SO(4)$. In fact the latter is additional structure, entering the theory as a parameter λ .

So let's get a feeling for what the 4-dimensional theory is by evaluating on a 4-manifold. So let M be a oriented closed 4-manifold. Then

$$\mathcal{A}_\tau(M) = \sqrt{\lambda|F|}^{\chi(M)} \mu^{\text{sign}\tau \cdot \text{sgn}(M)}$$

where μ is an 8th root of unity.

Under some circumstances, \mathcal{A}_τ is trivial on 4-manifolds. First one fixes λ such that the first factor vanishes. The other factor vanishes,

- (1) if $8|\text{sign}(\tau)$
- (2) if M is spin, $8|\text{sgn}(M)$

In the above cases we can choose a trivialization

$$\mathcal{A}_\tau \xrightarrow{T} 1$$

and compose

$$1 \xrightarrow{CS^\tau} \mathcal{A}_\tau \xrightarrow{T} 1$$

can be regarded as a 3-functor

$$\text{Bord}^3 \xrightarrow{T \circ CS^\tau} 2 - \text{Alg}.$$