1. Chern-Simons as a 3-2-1 Theory, Hiro Tanaka, Northwestern

The goal of this talk will be to introduce Chern-Simons theory, which is a 3-2-1 TFT. In the end, I'd like to comment about its relation to the 2-1-0 TFT given by $K_G(G)$, which Constantin talked about yesterday.

1.1. **Ed Witten's Fairy Tale.**

In 1989, Ed Witten told a fairy tale.[1] The main characters of this fairy tale were a 3-manifold M, and a simple, simply connected group G (think $SU(2)$, as usual), and a trivial principle G-bundle over M. If we're given an auxiliary player, a connection $A \in \Omega^1(M;\mathfrak{g})$, we can compute the number

$$CS(A) = \frac{1}{8\pi^2}\int_M tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

So far this isn't such a fantastic tale, and our children would be very bored listening to this fairly tale. Now, this gives us an invariant for a manifold M and a connection A. Is there a way to remove the dependence on A and get a *topological* invariant of M? Yeah. Let's do something crazy: let's integrate out the dependence on A by integrating over the space of all connections

$$Z_k(M) = \int_{\mathcal{A}_M/G^M} e^{2\pi i k CS(A)}.$$

Here, $\mathcal{A}_M$ is the space of connections. Note that I'm modding out by the action of some group—G^M denotes $C^\infty(M,G)$, the space of gauge transformations. This is because we don't want to double count two connections which are related by an automorphism of the trivial bundle. An element of the gauge group $g \in C^\infty(M,G)$ acts on a connection by

$$g^*A = g^{-1}Ag + g^{-1}dg.$$

Exercise 1. Show that

$$CS(g^*A) = CS(A) + n, \quad n \in \mathbb{Z}.$$

In the case that G is $SU(2)$, show that $n = deg(g)$. Note this implies that the integrand is well-defined on the space we're integrating over. Note also that Z_k depends on a choice of some integer k, so we get a number for each k.

Now this story's really a fairy tale! There's no measure on the space of connections mod gauge transformations, and as far as we're concerned, this partition function $Z_k(M)$ is a witch. It's a magical thing with seemingly no real root in reality. But most fairy tales probably came from some grain of truth: The magic witch with the poison apple was maybe just a bitter lady with some arsenic.

Witten claimed that we can define some 3-dimensional TFT from this data. The rest of the talk is devoted, then, to finding the arsenic behind this witch, and filling in the following table:

[1]This phrase, fairy tale, does a horrible injustice to the amount or real mathematics contained in Witten's 1989 paper. If you like, you can replace the words 'fairy tale' with the word 'physics' everywhere in this talk.

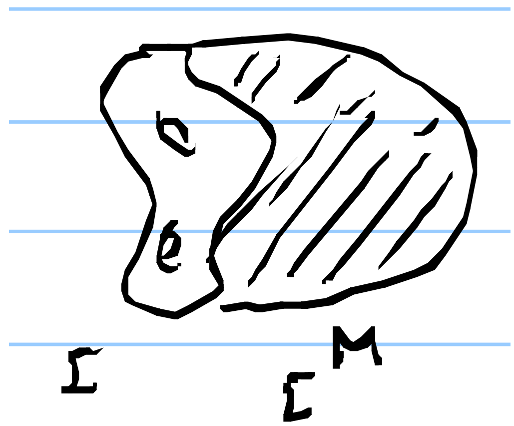

FIGURE 1. A 3-manifold M with boundary Σ.

$$\begin{array}{c|c} 3-\text{manifold } M & \text{partition function} \\ 2-\text{manifold} & ? \\ 1-\text{manifold} & Rep(\widetilde{LG}),\ \text{a linear category} \end{array} \tag{1}$$

I've given away the bottom row just in case I don't have time. But there's going to be a theme here, coming purely from physics: *given a manifold M with boundary, we want to study the space of connections on M which restrict to a particular connection on ∂M.* This will give us enough intuition to take a stab at what surfaces an circles should be assigned by Z_k.

1.2. **What Chern-Simons Assigns to a Surface.** Now let's figure out what to assign to a surface. So say we cut our 3-manifold, leaving a 3-manifold with boundary ${}_\Sigma M$. I'm using the notation ${}_\Sigma M$ to mean that M has an incoming boundary Σ, and no outgoing boundary. Let $\mathcal{A}_\Sigma$ be connections on Σ, fix some $a \in \mathcal{A}_\Sigma$. Following the theme, I am particularly interested in the following subset of $\mathcal{A}_M$:

$$\mathcal{A}_a := \{A \in \Omega^1(M; \mathfrak{g}) | A|_\Sigma = a\}$$

Because, you know what? I can get a number out of this! We can assign to the pair (M, a) the following number,

$$Z_k(M)(a) = \int_{\mathcal{A}_a/ker(G^M \to G^\Sigma)} e^{2\pi i k CS(A)}.$$

Exercise 2. Show that $\forall g \in G^\Sigma, a \in \mathcal{A}_\Sigma$, the function $Z_k(M)$ satisfies

$$Z_k(M)(g^*a) = e^{2\pi i k f(g,a)} Z_k(M_\Sigma)(a)$$

where the function f is defined by

$$f : \mathcal{A}_\Sigma \times G^\Sigma \to \mathbb{R}/\mathbb{Z}$$

$$(a, g) \mapsto \frac{1}{8\pi^2} \int_\Sigma tr(g^{-1} a g \wedge g^{-1} dg) - \int_M \widetilde{g}^* \sigma.$$

(Here, $\widetilde{g} \in G^M$ such that $\widetilde{g}|_\Sigma = g$, and $\sigma \in H^3(M;\mathbb{Z})$, where $\sigma[M] = n$ in the n above in the indeterminacy of the CS action when we change by a gauge transformation.)

Man, Hiro, you've wasted our time! You just got us another number that depends on a connection! Calm down, child. The punchline of this exercise is that if G^Σ acts on $\mathbb{C} \times \mathcal{A}_\Sigma$ by the action

$$g(z,a) = (e^{2\pi i f(g,a)}z, g^*a)$$

then we can consider the trivial bundle

$$\mathbb{C}^{\otimes k} \times \mathcal{A}_\Sigma \to \mathcal{A}_\Sigma,$$

because on this line bundle, $Z_k(M_\Sigma)$ is a G^Σ-equivariant section! Perhaps inspired by Borel-Weil-Bott theory, we love to study sections of bundles equivariant under some group action. In fact, maybe it is the vector space of sections of this line bundle that we should be assigning to Σ.

This takes us into a little digression, into the idea of geometric quantization.

1.2.1. *Geometric Quantization.* Say $L \to X$ is a complex line bundle over a Kähler manifold X with a G action, and $c_1(L) = \omega$ for ω a Kähler form. If G preserves ω, we may define a moment map,

$$\mu : X \to \mathfrak{g}^*$$

where $\xi \in \mathfrak{g}$ and $v \in T_xX$. This moment map is G-equivariant, and is defined by the identity

$$< d\mu_X(Y), \xi >= \omega_X(v, \xi_x).$$

Since the co-adjoint action on the Lie algebra's dual fixes 0, there is a G action on the space $\mu^{-1}(0) \subset X$.

Definition 1.1. $\mu^{-1}(0)/G$ is called a symplectic quotient of X.

So just to give a sense of where we're headed, $\mathcal{A}_\Sigma$ turns out to be a symplectic manifold and μ evaluates the curvature of a connection. The group action is of course the gauge group, so $\mu^{-1}(0)/G$ is the space of flat connections mod gauge transformation.

As a remark, notice that it might seem that μ is only defined up to a constant, but for simple groups this ambiguity goes away.

Now we want to complexify some stuff. We need the following:

Theorem 1.2. *G-equivariant sections of L are in bijection with holomorphic sections of $L/G_\mathbb{C} \to X/G_\mathbb{C}$. Further, if we do the symplectic reduction or the quotient by the complexification, we get the same manifold.*

So then we see that to a surface in Chern-Simons theory we associated to a surface, Γ_{hol}(flat connections on Σ/Gauge;$\mathcal{L}$), where the sections are holomorphic.

So how does this help us? By naturally considering connections on M that restrict to certain connections on Σ, we realized that we're studying G^Σ-equivariant sections of some line bundle. Moreover, when we see that we're studying holomorphic sections, we see that we have associated a *finite-dimensional* vector space to $\Sigma = \partial M$.

So we fill in our diagram:

$$\begin{array}{c|c} 3-\text{manifold } M & \text{partition function} \\ 2-\text{manifold} & \Gamma(\mu^0/G^\Sigma;\mathcal{L}) \\ 1-\text{manifold} & Rep(\widetilde{LG}) \end{array}$$

But I'd given away the punchline for a 1-manifold! Let me give some explanation as to why we should expect the representations of a loop group to show up.

1.3. **What Chern-Simons Assigns to a Circle.** So now let's explain what we assign to the circle. Let's consider a bordism Σ from the circle to the empty set . Then $S^1 \stackrel{\Sigma}{\to} \emptyset$ should gives us a map of *categories*

$$Z_k(S^1) \stackrel{Z_k(\Sigma)}{\to} Z_k(\emptyset).$$

Here we know what the empty set should be—it should be the linear category of vector spaces. This is based on the *monoidal structure* on $\mathbb{C}$-linear categories. We've run into a few questions about what this monoidal structure should be, so let me review it quickly here.

1.3.1. *The Monoidal Structure on $\mathbb{C}$-linear Categories.* What is the monoidal structure? Given two linear categories $\mathcal{C}$ and $\mathcal{D}$, we form their tensor product, written $\mathcal{C} \otimes \mathcal{D}$, in two steps. We can first form a category consisting of objects which are pairs (C, D) and whose morphisms are defined by

$$Hom((C,D),(C',D')) = Hom(C,C') \otimes Hom(D,D').$$

Often, the second step is to then take the (co)completion of this category by introducing enough colimits.[2] Finally, we also assume that the 2-category of linear categories has morphisms being colimit-preserving functors which are linear maps on each Hom-set.

Exercise 3. Show that the category of vector spaces, written Vect, is the unit for this tensor product.

So, just as we knew to assign $\mathbb{C}$ to the empty 2-manifold, we know to assign $Vect$ to the empty 1-manifold.[3]

Now we are done with preliminaries. Consider $\mathcal{A}_\Sigma$ as the space of connections on the trivial bundle over $G \times \Sigma \to \Sigma$, and G^Σ acts on $\mathcal{A}_\Sigma$ as before.

[2] There are two big reasons for this. For one thing, we may only want to consider categories which have small (or finite) limits and colimits. Second, we want the following to hold in general: For given algebras R and S, we want the category of $R \otimes S$-modules to be equivalent to the category $(R - mod) \otimes (S - mod)$. Clearly, not all $R \otimes S$-modules can be written as tensors of R-modules and S-modules, but they can be written as colimits of such.

[3] This is another point which came up in the talk. The empty manifold is a manifold of every dimension, since the dimension condition is vacuous.

We have a symplectic structure on the space $\mathcal{A}_\Sigma$ given by taking the trace of a wedge product and integrating:

$$\omega(\alpha, \beta) = \frac{1}{8\pi^2} \int_\Sigma tr(\alpha \wedge \beta).$$

We are identifying $\mathcal{A}_\Sigma$ with its own tangent space at every point, because it is a linear space.

Now given this symplectic structure and the action of the gauge group, we can make a moment map

$$\mu : \mathcal{A}_\Sigma \to (Lie(G^\Sigma))^*$$
$$\mu(A) = Curv(A) - \psi(A)$$

where ψ is a map which requires some explanation. It is the composition

$$\mathcal{A}_\Sigma \to \mathcal{A}_{S^1} \overset{A}{\hookrightarrow} (\widetilde{L\mathfrak{g}})^* \overset{B}{\to} (Lie(G^\Sigma))^*.$$

Here, B is the dual of the map of lie algebras obtained from lifting a map $G^\Sigma \to LG$ to $G^\Sigma \to \widetilde{LG}$ over LG, and taking the derivative. (We are using the obvious map from $G^\Sigma \to LG$ by restricting to the boundary circle.) The map A is given by the natural inclusion of $\Omega^1(S^1; \mathfrak{g})$ into the dual of the Lie algebra of the loop group, which I think Dario mentioned in his talk. (Clearly, given a Lie algebra element in the extension of the loop group, and a connection on the circle, we can pair them together as follows:

$$a(v) = \int_{S^1} \langle a, v \rangle.$$

This is the map from $\mathcal{A}_{S^1}$ into the dual of the loop algebra.)

Now, I'm supposed to give you an argument as to why $Z(S^1)$ should be the category of representations of a loop group. Let me tell you the argument now.

Fix a representation $\hat{R}$ of $\widetilde{LG}$. From Dario's talk, this determines a co-adjoint orbit in the dual of the loop algebra. By the map B from above, this gives a co-adjoint orbit W in $(Lie(G^\Sigma))^*$. Conversely, this co-adjoint orbit gives us connections on S^1 via A, and (up to the gauge group action) this determines a conjugacy class $C \subset G$ by evaluating the holonomy around the circle. Here is a big theorem:

Theorem 1.3. *Fix an irreducible representation $\hat{R}$ of $\widetilde{LG}$. Then*

$$\mu^{-1}(W)/G^\Sigma = \{\text{cong classes of } \pi_1(\Sigma) \to G \text{ s.t. } \pi_1(\partial\Sigma) \to C\}$$

where W is the image of the coadjoint orbit in $(\widetilde{LG})^*$ associated to $\hat{R}$ and C is a conjugacy class in G given by the holonomy of the LG-orbit in $\mathcal{A}_{S^1}$ that come from the $\widetilde{LG}$-orbit $(\widetilde{LG})^*$.

Remark: this is closely related to the Kirillov story, but it is "holomorphic, not Dirac," so there are some minor differences.

Now let's define

$$\mathcal{M}_{\Sigma;C} = \mu^{-1}/G^\Sigma.$$

This giant theorem continues:

Theorem 1.4. *$\mathcal{M}_{\Sigma;C}$ is Kähler and we pick a line bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = \omega$. Then the space of sections of $\mathcal{L}^{\otimes k} \to \mathcal{M}_{\Sigma;C}$ gives a functor*

$$Z_k(S^1) \to Vect$$
$$\hat{R} \mapsto \Gamma(\mathcal{M}_{\Sigma;C}, \mathcal{L}^{\otimes k})$$

which is the one we want.

So we went through a lot of mathematics just now, but I hope this motivates why one would expect Chern-Simons to assign $Rep(\widetilde{LG})$ to the circle.

1.4. **Relationship to Twisted K-Theory.** Here is a theorem, which I think was first proven by Verlinde. Witten mentions it in his 1989 paper and references Verlinde, but without any mention of the *algebra structure*, only of the vector space.

Theorem 1.5. *$Z_k(S^1 \times S^1) =$Verlinde algebra for $\widetilde{LG}$.*

Remark 1.6. This shouldn't surprise anybody, if we have the general philosophy that $X \times S^1$ should always get the Hochschild homology of X. This is because the Verlinde algebra is a vector space with a very nice basis, given by irreducible representations of the loop group at level k. This is completely analogous to the situation for the Hochschild homology of the category of G-representations, or the category of R-modules. (Constantin had a very interesting comment about this involving bundles on elliptic curves, but it was lost on me.)

One should note that the multiplication on this algebra, as dicussed by AJ, is not tensor product. Tensor products do not preserve a level k. Instead, it is the *fusion product.*

Now, you might see the connection to twisted K-theory. Here is the famous theorem due to Freed, Hopkins, and Teleman.

Theorem 1.7. *The Verlinde algebra and ${}^{\tau-d}K_G(G)$ are isomorphic rings, where the multiplication on K-theory is given by the Pontrjagin product.*

The relationship between CS assigning the Verlinde algebra to the torus and ${}^{\tau}K_G(G)$ assigning the Verlinde algebra to a circle is not a conicindence: It turns out we can see it from dimensional reduction.

1.4.1. *Dimensional Reduction.* Let Cob_n^{n+1} be the category whose objects are n-manifolds and whose morphisms are $(n+1)$-manifolds. (During my talk, I took the example of $n = 0$ to make things easy.) If we fix some $(n-k)$ manifold M, we see there is a functor

$$\times M : Cob_k^{k+1} \to Cob_n^{n+1}.$$

which for every object and cobordism, just takes the product with M. Hence, given a TFT

$$Z : Cob_n^{n+1} \to Vect$$

we can pre-compose with the functor $\times M$ to obtain a new TFT

$$Z' : Cob_k^{n+1} \to Vect.$$

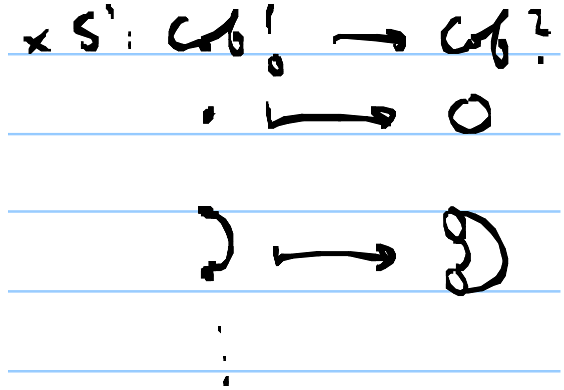

FIGURE 2. Dimensional reduction.

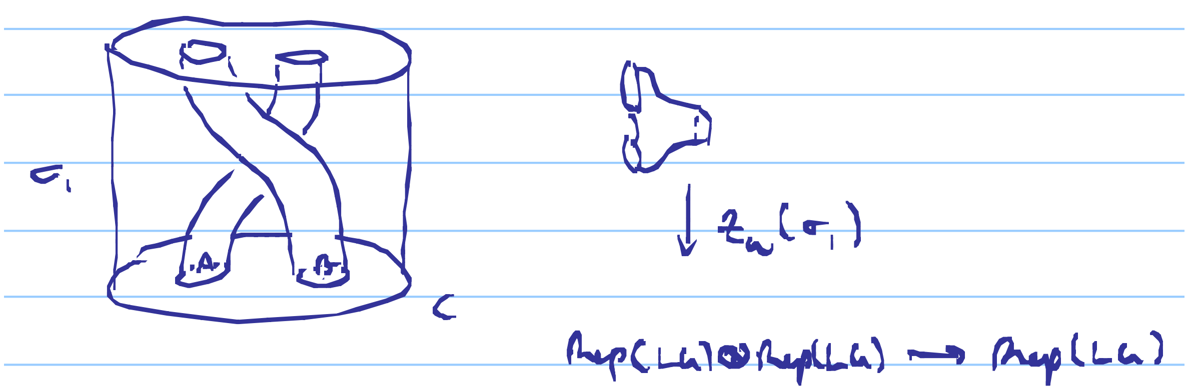

FIGURE 3. Braiding coming from 3-d geometry.

This same discussion applies to extended TFTs—for instance, to 3-2-1 TFTs as opposed to 3-2 TFTs.

Definition 1.8. A TFT Z' obtained in this way is called a *dimensional reduction of* Z.

So here is the big connection between Chern-Simons theory and twisted K-theory:

Proposition 1.9. *The 2-1 TFT* ${}^{\tau}K_G(G) \otimes \mathbb{C}$ *is dimensional reduction by* S^1 *of the 3-2 TFT given by CS.*

The conjecture is that this extends down to points. That is, that the 2-1-0 TFT is the dimensional reduction of the 3-2-1 Chern-Simons theory.

But there is a huge piece of structure that is lost in dimensional reduction. The circle had a pair of pants, where as the point certainly does not. This relates to the stuff Constantin was saying yesterday about what he called 'Dirac convolution,' so I'd like to talk about this structure now.

1.5. **Braided Monoidal Categories.** Now let's talk about braided monoidal categories. This is the structure that any category $Z(S^1)$ has in a 3-2-1 TFT.

Consider the squashed pair of pants (a disk with two holes removed). Then consider a cobordism, given by a 3-manifold, that "swaps the legs," see the picture. This is going to lead to a braiding. Now a pair of pants gives a functor

$$Rep(\widetilde{LG}) \otimes \widetilde{LG} \to \widetilde{LG},$$

and this braiding gives a natural transformation, which we will call σ_1. The appropriate tensor product here is the fusion product.

I'd like to contrast a braided monoidal category with a symmetric monoidal category $(\mathcal{C}, \otimes)$. In a symmetric monoidal category, there is the natural transformation which relates the functor

$$\mathcal{C} \times \mathcal{C} \overset{swap}{\to} \mathcal{C} \times \mathcal{C} \overset{\otimes}{\to} \mathcal{C}.$$

with the functor

$$\mathcal{C} \overset{\otimes}{\to} \mathcal{C}$$

and this natural transformation is required to be an isomorphism on objects.

However, in a braided monoidal category, the natural transformation given by the braid σ_1 need not have at all the structure of the natural transformation from a symmetric monoidal category! For instance, σ_1^2 is not the identity, whereas it is in a symmetric monoidal category. In fact, by drawing all the 3-cobordisms you might think of for a pair of pants, we see that there is an action of the braid group on the space of natural transformations from the pair of pants to itself. Such a category, along with the tensor product $\otimes$, is called a *braided monoidal category.*

So now we see the open question from yesterday: Is there a "Dirac convolution" on ${}^{\tau}Vect_G(G)$ that corresponds to the fusion product $Rep(\widetilde{LG})$?

Remark: for negative energy representations at negative levels, we get some dual picture, in Chern-Simons, but it doesn't give anything new.

Another remark: ${}^{\tau}Vect_G(G)$ is a categorification of ${}^{\tau}K_G(G)$. This is the extending-down problem, roughly stated this trying to find a product structure that works on the vector bundles themselves, not their image under K-theory.