Dirac family construction of K-classes

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1 Overview

In this lecture we discuss three similar constructions, giving a (twisted) K-theory class from a irreducible representation. Heuristically, one can see this as a concrete and robust implementation of the Thom isomorphism or as an inverse to a Kirillov correspondence. We will discuss the following three cases:

G. If G is a compact Lie group, then let $R(G)$ be the representation ring of complex finite dimensional representations. Then there is an isomorphism:

$$R(G) \to {}^{\sigma}K_G^{\dim G}(\mathfrak{g}^*)_{cpt}$$

LG. This is the generalization of the situation for G. Let τ be an admissible central extension of LG (i.e. a twist). Let $R^\tau(LG)$ denote representation ring of positive energy representations at level τ. Then there is an isomorphism:

$$R^{\sigma-\tau}(LG) \to {}^{\tau}K_G^{\dim G}(G)$$

LT. Let T be a torus and let τ be an admissible central extension of LT (i.e. a twist). Then positive energy representations of LT at level τ are in one-to-one correspondence with representations of the finite-dimensional, but non-compact Heisenberg-type group Γ^τ and more precisely the representation rings $R^\tau(LG)$ and $R(\Gamma^\tau)$. Then there are isomorphisms:

$$R^\tau(LG) \to R(\Gamma^\tau) \to {}^{\tau}K_T^{\dim T}(T)$$

All these isomorphisms are implemented by an equivariant family of Dirac operators on spinor fields, indexed by spaces $\mathfrak{g}^*$, $\mathfrak{t}$, $\mathcal{A}$ respectively.

2 Special twistings

We have seen the central extension of groupoids model of a twisting. All twistings appearing in this lecture will be a special instance of these twistings.

We will twist ${}^{\tau}K_G^*(X)$ by twists of the following type. We have a local equivalence $\tilde{X}//\tilde{G} \to X//G$ induced by the quotient map with respect to a normal subgroup $\tilde{N}$ of $\tilde{G}$ that acts freely on $\tilde{X}$ such that $\tilde{G}/\tilde{N} = G$ and $\tilde{X}/\tilde{N} = X$. Secondly we have a central extension $\tilde{G}^\tau$ of G and finally we have a homomorphism $\tilde{G} \to \mathbb{Z}/2\mathbb{Z}$ giving a grading.

Then the concrete K classes for this twist are given $\tilde{G}^\tau$ equivariant families of skew-adjoint odd Fredholm operators on a τ-twisted Hilbert bundle over $\tilde{X}$, possibly with commuting $Cl^c(1)$-action. If there is such a commuting $Cl^c(1)$-action, we get an odd K-theory element. If it is not there, we get an even K-theory element.

3 Pin^c and spinors

Let V a real vectorspace with inner product. The dual V^* with dual inner product gives us a complex Clifford algebra $Cl^c(V^*)$ This contains as a subgroup $Spin^c(V)$, which is a central extension of $SO(V)$ by $\mathbb{T}$:

$$1 \to \mathbb{T} \to Spin^c(V) \to SO(V) \to 1$$

This extension can be generalized to $O(V)$, giving a group $Pin^c(V) \subset Cl^c(V^*)$ which is a central extension of $O(V)$ by $\mathbb{T}$:

$$1 \to \mathbb{T} \to Pin^c(V) \to O(V) \to 1$$

This gives a short exact sequence of Lie algebras:

$$0 \to i\mathbb{R} \to \mathfrak{pin}^c \to \mathfrak{o} \to 0$$

and this is canonically split by a map $\mathfrak{o} \to \mathfrak{pin}^c$.

The representation theory of $Cl^c(V^*)$ is pretty easy (see [Pre86, chapter 12]). We consider $Cl^c(V^*)$ as a $\mathbb{Z}/2\mathbb{Z}$-graded algebra with graded representations. If $\dim V$ is even, there exist two irreducible $\mathbb{Z}/2\mathbb{Z}$-graded representations, which are isomorphic up to a graded shift. We fix one. If $\dim V$ is odd, there exists one irreducible representation, which allows a commuting $Cl^c(1)$-action. We will keep track of this action.

For any value of $\dim V$, we can therefore fix a irreducible graded spinor representation S, with a commuting $Cl^c(1)$-action if $\dim V$ is odd.

In the infinite dimensional case, working with a Hilbert space H, this theory works similarly, expect that one needs to fix a polarization of the Hilbert space first. Roughly one needs to seperate positive from negative energy vectors.

4 The theorem for G

There is a canonical central extension G^σ of G. Consider the adjoint representation $Ad : G \to O(\mathfrak{g})$. We can use this to form the pullback $G^\sigma : Pin^c(\mathfrak{g}) \times_{O(\mathfrak{g})} G$. This acts on the spinor representation S of $Pin^c(\mathfrak{g})$ through projection. We denote the infinitesimal action of $\mathfrak{g}$ through the splitting $\mathfrak{pin}^c$ using the splitting by σ. There is an additional Clifford multiplication action of $\mathfrak{g}^*$, which we denote by γ.

Fix a basis e_a of $\mathfrak{g}$ and denote the dual basis by e^a. Then we denote the action of e_a through spinor multiplication by σ_a and the action of e^a through Clifford multiplication by γ^a.

If we trivialize the cotangent bundle of G by left translation, then the spinor fields (obtained by the construction explained earlier, by fiberwise constructing an irreducible $Cl^c(T^*_g G)$ representation) can be identified with $C^\infty(G) \otimes S$. The infinitesimal version of the left translation action of $C^\infty(G)$ gives another action R of $\mathfrak{g}$. We denote the action of e_a by R_a.

So we end up with a $C^\infty(G) \otimes S$ admitting three actions: σ by $\mathfrak{g}$, R by $\mathfrak{g}$ and γ by $\mathfrak{g}^*$. This allows us to define the Dirac operator family D:

Definition 4.1 (The Dirac operator family D)**.** Define $D : \mathfrak{g}^* \to End(C^\infty(G) \otimes S)$ as follows:

$$\mu = \mu_a e^a \mapsto D_\mu = i\gamma^a R_a + \frac{i}{3}\gamma^a \sigma_a + \mu_a \gamma^a$$

Each D_μ^2 is indeed a generalized Laplacian, so this is a Dirac operator in the analytic sense. It is a skew-adjoint odd G^σ-equivariant operator and if S admits a commuting $Cl^c(1)$-action, D_μ commutes with this for each $\mu \in \mathfrak{g}^*$.

However, $C^\infty(G) \otimes S$ is highly reducible, so we restrict our attention to smaller subspaces. To do this, we use the Peter-Weyl theorem:

Theorem 4.2 (Corollary of Peter-Weyl). *$L^2(G) \otimes S$ decomposes as follows:*

$$L^2(G) \otimes S \cong \bigoplus_{V \in \hat{G}} V^* \otimes V \otimes S$$

where $\bigoplus$ denotes the completed tensor product, $\hat{G}$ the set of irreducible representations of G, G acts on V^* by right translation, on V by left translation and projective on S through G^σ.

By the G^σ-equivariance of the Dirac family, we can restrict it to $V^* \otimes V \otimes S$ and because it doesn't involve right translation, we can forget about the factor V^*. We denote the Dirac family restricted to $V \otimes S$ by $D(V)$.

It is easily seen now, since $V \otimes S$ is finite dimensional, that is a G^σ-equivariant family of skew-adjoint odd Fredholm operators with commuting $Cl^c(1)$-action if $\dim \mathfrak{g}$ is odd. We claim that it is compactly supported over the coadjoint orbit of $-\lambda - \rho$, where $-\lambda$ is lowest weight of V and $-\rho$ with $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$ is the lowest weight of S as a G^σ-representation, where Δ^+ denotes the set of positive roots. This is proven by calculating the character and the dimension of S and using the Weyl character and dimension formulae to link S to the standard representation of lowest weight $-\rho$.

To do the calculation of the kernel of D_μ for a fixed $\mu \in \mathfrak{g}^*$, we pick a maximal torus T_μ and a weight-space decomposition such that μ is antidominant.

Proposition 4.3. *If V is irreducible, then $\ker D_\mu$ is $V_{-\lambda} \otimes S_{-\rho}$ is $\mu = -\lambda - \rho$.*

Sketch of proof. One defines an energy operator E_μ which behaves as multiplication by a positive constant on weight space and achieves its lowest value on the lowest weight space $V_{-\lambda} \otimes S_{-\rho}$. From a simple calculation is follows that $D_\mu^2 + 2E_\mu$ is multiplication by a fixed constant on the entire space $V \otimes S$. Therefore D_μ^2 is maximal on the lowest weight space. A calculation of its value of $V_{-\lambda} \otimes S_{-\rho}$ then shows that is non-positive there and zero if and only if $\mu = -\lambda - \rho$. □

This implies that we get a map from representations to twisted K-theory, where the twist is $(\mathfrak{g}^*//G \to \mathfrak{g}^*//G, G^\sigma, \epsilon^\sigma)$.

Theorem 4.4. *The map $\Psi : R(G) \to {}^\sigma K^{\dim G}(\mathfrak{g}^*)_{cpt}$ is a implementation of the twisted Thom isomorphism.*

Proof. It is homotopic through compactly supported Fredholm operators to the standard implementation of the Thom isomorphism, $[V] \mapsto (V \otimes S, id \otimes \gamma)$. □

One can generalize these constructions to projective representations of G with cocycle τ, i.e. representations of a central extension G^τ of G. Then $\mathfrak{g}^*$ is replaced by the $\mathfrak{g}^*$-torsor $\mathcal{A}$ of connections, which are in bijection with splittings $\mathfrak{g}^\tau \to \mathfrak{g}$. These are necessarily to define R_a, which now depends on $\mu \in \mathcal{A}$.

5 The theorem for LG

This theorem is closely related to the case of projective representations of G. The construction is very similar: one defines a family of Dirac operators to implement a map from irreducible positive energy representations to twisted K-classes.

The main differences with the latter cases will be:

- Analysis gets involved because $L\mathfrak{g}$ is infinite dimensional. For exmaple, one needs to work on dense subspaces of positive energy vectors. Similarly, there are some problems with naive definitions of Pin^c and the spinor representation S. These now depend on a polarization J.
- Positive energy representations are projective, i.e. they are representations for some central extensions $(LG)^\tau$ of LG. This requires the use of connections or (under suitable conditions) equivalently splittings $(L\mathfrak{g})^\tau \to L\mathfrak{g}$.

Let τ be an admissible central extension of LG. Roughly speaking, it means that the central extension extends to $\tilde{L}G = \mathbb{T} \ltimes LG$ and that there is a nice inner product on $(\tilde{L}\mathfrak{g})^\tau$. For semisimple G every central extension is admissible. Let V be a positive energy representations of level $\tau - \sigma$. Then an infinite dimensional spinor representation of $Cl^c((L\mathfrak{g})^*)$ exists.

Let $e_a z^n$ be a Fourier basis of $L\mathfrak{g}^*$. Then $\gamma^a(n)$ denotes the Clifford multiplication on S, $\sigma_a(n)$ the spinor action of S and $(R_a(n))$ the infinitesimal action of $e_a z^n$ on V with respect to connection $A \in \mathcal{A}$ (this uses the splitting $L\mathfrak{g} \to (L\mathfrak{g})^\tau$ coming from the connection).

Definition 5.1 (Dirac family)**.** We define $D_A = i\gamma^a(-n)(R_a(n))_A + \frac{i}{2}\gamma(\Omega)_A$ as our Dirac family depending on a connection $A \in \mathcal{A}$. If we fix a connection A_0, e.g. the trivial one, then this formula simplifies to $D_\mu = i\gamma^a(-n)R_a(n) + \frac{i}{3}\gamma^a(-n)\sigma_a(n) + \gamma(\mu)$ for $\mu \in (L\mathfrak{g})^*$.

However, D_μ involves differentation, so in general will not be bounded. Therefore, we replace it by $F_A = D_A(1 - D_A^2)^{-1/2}$. One can then show, using the same techniques as for the case of G, that F_A is a $(LG)^\tau$-equivariant family of skew-adjoint odd Fredholm operators on $\mathcal{A}$, with commuting $Cl^c(1)$-action if $\dim G$ is odd.

Thus we get an element in ${}^\tau K_G^{\dim G}(G)$ with τ the twist given by $(\mathcal{A}//LG \to G_1//G, (LG)^\tau, \epsilon)$, where G_1 is the identity component of G. This follows from the fact that the holonomy only takes values in G_1 for a trivial G-bundle over the circle.

Theorem 5.2. *The map $\Psi : R^\tau(LG) \to {}^\tau K_G^{\dim G}(G_1)$ given by this Dirac family construction is an isomorphism.*

This generalizes almost directly in two directions: gauge groups for any principal G-bundle P over S^1 (then we have to replace G_1 with $G[P]$, the image of the holonomy) and fractional loops (then we replace the rotation circle by a finite cover).

6 The theorem for LT

A simple case of the main theorem of [FHT07] is that of tori. It has the virtue of being reasonably hands-on and it is also of importance in the proof of the theorem.

It is easy to prove LT is isomorphic to a product $\Gamma \times \exp(V)$ of the group $\Gamma = \Pi \times T$ of loops of which the projection of the velocity to $\mathfrak{t}$ is constant and the exponential of the subspace of the Lie algebra $L\mathfrak{t}$ complementary to $\mathfrak{t}$ (one can introduce the L^2-product one also uses for the Fourier decomposition of loops).

Let Π denote the character lattice and Π^* the cocharacter lattice. A homomorphism $\tau : \Pi \to \Pi^*$ can be considered as an element of $H^1(T) \otimes H_T^2(T) \cong H^1(T) \otimes H^1(T)$ and hence as a twist. If it is of full rank, it is admissible and the central extension $(LT)^\tau$ splits as $\Gamma^\tau \times \exp(V)^\tau$. The former is a group of Heisenberg-type.

The irreducible representations of $\Gamma^\tau \times \exp(V)^\tau$ are easily classified. The group $\exp(V)^\tau$ admits a single irreducible positive energy representation, the Fock representation $\mathcal{F}$. The Heisenberg-type group admits infinite dimensional representations $V_{[\lambda]} = \bigoplus_{\lambda \in [\lambda]} V_\lambda$ indexed by $[\lambda] \in \Pi^*/\tau(\Pi)$. From this we see that $L^2(T) \otimes S \cong \hat{\bigoplus} V_{[\lambda]}$.

This discussion shows that positive energy representations of LT at level τ are in bijection with representations of Γ^τ. We will link the latter to K-classes using the Dirac family described before.

The map $\tau : \Pi \to \Pi^*$ induces a map $\tilde{\tau} : \Pi \to \Pi^*$. Therefore, the Dirac operator family $D_\mu = i\gamma^a R_a + \mu_a \gamma^a$ ($\sigma_a = 0$ since it depends linearly on the structure constants of the Lie algebra $\mathfrak{t}$) on $\mathfrak{t}^*$ can be pulled back to a give a Dirac family $\tilde{\tau}^* D$.

In fact, $\tilde{\tau}^* D$ turns out to be a Γ^τ-equivariant family of skew-adjoint odd Fredholm operators on $\mathfrak{t}$, with commuting $Cl^c(1)$-action if $\dim \tau$ is odd. Thus we obtain an element of ${}^\tau K_T^{\dim T}(T)$ where the twist is $(\mathfrak{t}//\Gamma \to T//T, \Gamma^\tau, \epsilon^\tau)$.

Theorem 6.1. *The map $\Psi : R(\Gamma^\tau) \to {}^\tau K_T^{\dim T}(T)$ given by this Dirac family construction is an isomorphism.*

In fact, using the isomorphism $K_T^{\dim T}(T) \cong Hom_{W^e_{aff}}(\Lambda^\tau, H^n_c(\mathfrak{t}) \otimes \mathbb{Z}[\epsilon])$, one can show that $[\lambda]$ is mapped to a function supported on $[\lambda]$ after identifying the Π^*-torsor Λ^τ with Π^*.

References

[FHT07] D.S. Freed, M.J. Hopkins, and C. Teleman. Loop groups and twisted k-theory ii. 2007. arxiv:math/0511232v2.

[Pre86] S. Pressley, A. & Segal. *Loop Groups*. Oxford University Press, 1986.