Dirac family construction of K-classes

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1 Dirac family construction for tori

1.1 Tori and spectral flow

We will define K-theory classes by providing families of Dirac operators over the dual torus or the torus, giving a Fredholm family. These will be simple cases of the construction of a Dirac family for a compact group, where the fact that the torus is abelian simplifies the construction.

1.1.1 K-theory and spectral flow

Let T be a torus with Lie algebra $\mathfrak{t}$ and choose a metric on $\mathfrak{t}$. Let e_a denote a basis of $\mathfrak{t}$ and let Π denote the integer lattice in $\mathfrak{t}$, so that we can identify T with $\mathfrak{t}/\Pi$. Note that $\Pi \cong \pi_1(T)$. Let $\mathfrak{t}^*$ denote the dual of the Lie algebra with dual metric and dual basis e^a. Let Π^* denote the dual lattice $Hom(\Pi, \mathbb{Z})$.

The metric on $\mathfrak{t}$ allows us to define the Clifford algebra $Cl^c(\mathfrak{t}^*)$. Let S be an irreducible spinor representation, if $\dim T$ is odd with commuting $Cl^c(1)$-action. We will denote the Clifford action by γ, and in particular Clifford multiplication by e^a with γ^a.

We will consider the spinor fields $L = C^\infty(T) \otimes S$. This has two actions: T acts by translation on $C^\infty(T)$, hence $\mathfrak{t}$ infinitesimally by differentiation $R_a := -i\frac{\partial}{\partial\theta^a}$, and $\mathfrak{t}^*$ by Clifford multiplication on S.

Definition 1.1. Let D be the family of operators $D : \mathfrak{t}^* \to End(L)$ given by

$$\mu = \mu_a e^a \mapsto D_\mu = i\gamma^a R_a + \mu_a \gamma^a$$

Note that for any $\mu \in \mathfrak{g}^*$, D_μ is an odd skew-adjoint operator. There is a special property of D_μ with respect to multiplication by characters of T: they give a translation action of Π^2 on the family of Dirac operators. Every $\lambda \in \Pi^*$ gives us a character $\chi_\lambda : T \to \mathbb{C}^*$ using the following construction: λ extends to a map $\mathfrak{t} \to \mathbb{R}$, which maps the integral lattice to integers. Using surjectivity of exp, we set $\chi_\lambda(\exp(X)) = \exp(2\pi i \chi_\lambda(X))$ and this is well-defined since the inegral lattice is mapped to the integers.

Definition 1.2. For $\lambda \in \Pi^*$, let $M_\lambda : L \to L$ be the operator given by $f(t) \mapsto \chi_\lambda(t)f(t)$.

Proposition 1.3. $D_\mu M_\lambda = M_\lambda D_{\mu+\lambda}$.

Proof. You know how to differentiate, right? □

This implies we can factor D has a family over T. Consider the quotient bundle $\mathfrak{t}^* \times_{\Pi^*} L$ where Π^* acts on $\mathfrak{t}^*$ by translation and on L by the M_λ. This is a bundle over $\mathfrak{t}^*/\Pi^* =: T^*$, the dual torus.

Since any Hilbert bundle is trivial by Kuiper's theorem, we therefore get a family of skew-adjoint Fredholm operators over T^*. If $\dim T$ is odd, we have a commuting $Cl^c(1)$-action. Therefore we get a class in $K^{\dim T}(T^*)$. To see this, we don't even need the technology of [FHT07a]. Because the bundle of Fredholm operators is trivial, Atiyah-Singer's results are enough.

The link with spectral flow, as defined in Atiyah-Patodi-Singer, is as follows. In the case of the circle, i.e. $\mathbb{T} = S^1$, the operator $D_\mu : L \to L$ has spectrum $\{i(n+\mu) | n \in \mathbb{Z}\}$ and thus one eigenvalues passes from the positive to the negative imaginary eigenspaces if we cross an integer. On $\mathbb{R}/\mathbb{Z}$, the analog of T^*, this phenomena is known as spectral flow: walking in a circle makes some eigenvalues pass 0. This number is independent of continuous deformations of the family of Fredholm operators.

1.1.2 Heisenberg groups and Heisenberg-type groups

To each locally compact topological abelian group T we can assign a Heisenberg group as follows: let $\hat{T} = Hom(T, \mathbb{T})$ by the Pontryagin dual of T. Then consider the group H_T of elements $T \times \mathbb{T} \times \hat{T}$ with multiplication $(t_1, \theta_1, \xi_1)(t_2, \theta_2, \xi_2) = (t_1 + t_2, \theta_1 + \theta_2 + \xi_1(t_2), \xi_1 + \xi_2)$.

Remember that $L^2(T) \cong L^2(\hat{T})$ using Fourier transform. Then H_T acts on both: on the $L^2(T)$, T acts by translation, $\mathbb{T}$ by scalar multiplication M_λ and $\hat{T}$ by multiplication with the associated character and on the $L^2(\hat{T})$, T acts with the associated character, $\mathbb{T}$ by scalar multiplication and $\hat{T}$ by translation. The Fourier transform intertwines these two actions. Infinitesimally, we get an action on $L^2(T)$ (or $L^2(\hat{T})$) of multiplication and differentation satisfying the canonical commutation relations (as in quantum mechanics).[1]

Now, let's look at Heisenberg-type groups. These depend on a homomorphism $V \to \hat{T}$, which is used centrally extend $T \times V$ as the smallest subgroup of H_T containing T, $\mathbb{T}$ and V as elements. In the case of T our torus, $\hat{T} \cong \Pi^*$, and we can get Heisenberg type groups from H_T^τ from a linear map $\tau : \Pi \to \Pi^*$. We can describe the group H_T^τ explicitly as the central extension of $\Pi \times T$ defined using the following commutation rule, where $p \in \Pi$, $t \in T$:

$$ptp^{-1} = \chi_{\tau(p)}(t)t$$

1.1.3 Twisted K-theory and spectral flow

We will now prove a theorem which links twisted equivariant K-theory of the torus with irreducible representations of a Heisenberg-type group derived from the torus.

Lemma 1.4. *A map $\tau : \Pi \to \Pi^*$ represents a twist of $K_T^*(T)$, either as an element of $H_T^3(T)$, or by giving a central extension of a group.*

If $\tau : \Pi \to \Pi^*$ has full rank, i.e. is an isomorphism of vector spaces after tensoring with $\mathbb{Q}$, there are finitely many unitary irreducible representations of H_T^τ, which are indexed by an equivalence class $[\lambda]$ in $\Pi^*/\tau(\Pi)$. These are infinite dimensional and occur once in the L^2-completion of $C^\infty(T) \otimes S$. We denote these by $F_{[\lambda]}$ and by definition $F_{[\lambda]}$ is exactly the completion of the direct sum of the weight spaces of T in $C^\infty(T) \otimes S$ for all weights in in the equivalence class $[\lambda]$. In fact, this is quite easy to prove. The hard part is showing that these are all unitary irreducibles. But we won't need the latter statement, so we just prove the easy first statement.

Proposition 1.5. *The subspace $F_{[\lambda]} = \bigoplus_{\lambda \in [\lambda]} F_\lambda$ of weight spaces F_λ for the T-action on the L^2-completion of $C^\infty(T) \otimes S$ is an irreducible unitary H_T^τ-representation with respect to the L^2-inner product.*

Proof. The definition of the action of the T, $\mathbb{T}$ and $\hat{T}$ easily shows that the action is unitary.

Note that the action of T and $\mathbb{T}$ preserves the weight spaces, and action of M_p sends the weight space of weight λ isomorphically to the weight space of weight $\lambda + \tau(p)$. This proves that $F_{[\lambda]}$ is H_T^τ-invariant. To see that it is irreducible, note that weight spaces F_λ are one-dimensional and irreducible, using Peter-Weyl together with the fact that T is abelian. □

The map τ extends to a linear map $\mathfrak{t} \to \mathfrak{t}^*$ sending the integer lattice Π to the dual integer lattice Π^*. Thus there is an induced maps $\tilde{\tau} : T \to T^*$. We can take the pullback of the $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $\mathfrak{t}^* \times_{\Pi^*} L$ over T^* using this map. This splits as vector bundles $F_{[\lambda]} \otimes S$. The family D_ξ of Dirac operators obtained by pullback of the family of operators D_μ preserves this decomposition. In fact, it is invertible unless $\xi \in \exp(\tau^{-1}([-\lambda]))$.

To get more information from this construction, we lift the operators from a family over T to a family over $\mathfrak{t}$. These are Γ^τ-equivariant.

Proposition 1.6. *The family $D : \mathfrak{t} \to End(F_{[\lambda]} \otimes S)$ is a family of odd skew-adjoint Fredholm operators and is Γ^τ-equivariant.*

[1] Take $T = \mathbb{R}^n$, then $\hat{T} \cong \mathbb{R}^n$. In the case we can replace $\mathbb{T}$ by $\mathbb{R}$ to retain more information and we get the group which is commonly known as the Heisenberg group. In this case the Stone-von Neumann theorem tells up for each choice of Planck's constant $\hbar$, there is a unique unitary irreducible representation up to unitary equivalence and these are all unitary irreducible representations up to unitary equivalence. Note the contrast with the representation theory of our Heisenberg groups.

Proof. The first property is obviously preserved by pullback. The Γ^τ-equivariance is not much harder. Γ^τ acts on $F_{[\lambda]} \otimes S$ as by the action of Γ^τ on $F_{[\lambda]}$. On $\mathfrak{t}$ its action factors through Π, which acts by translation. The equivariance then follows from the relation $D_\mu M_\lambda = M_\lambda D_{\mu+\lambda}$ for the original Fredholm family. □

Note that since $\mathfrak{t}$ is a Π-principal bundle over T, the quotient map by Π gives a map $\mathfrak{t}//\Gamma \to T//T$ which is a local equivalence of groupoids. Furthermore Γ^τ is a central extension of Γ, therefore $(\mathfrak{t} \to T, \Gamma^\tau, \epsilon)$ gives a twist of equivariant twisted K-theory of T. This means that the Fredholm operators define a class in $K_T^{\tau+\dim T}(T)$. The twisted K-theory obtained in this manner form a basis of $K_T^{\tau+\dim T}(T)$, which means that the following theorem holds:

Theorem 1.7. *The construction of a twisted K-theory class from a representation $F_{[\lambda]}$ induces an isomorphism of abelian groups:*

$$\Psi : R(\Gamma^\tau) \to K_T^{\tau+\dim T}(T)$$

1.1.4 Decomposition of loop groups of tori

This section is a small preview, but not irrelevant since it appears in the proof of the theorem of section 4. We will describe how the theorem of the last section in fact describes isomorphism classes of positive energy representations of LG at the level τ.

We define two special subgroups of LG. The Lie algebra $\mathfrak{g}$ has a center $\mathfrak{z}$. Let $L\mathfrak{z}$ denote the Lie subalgebra of $L\mathfrak{g}$ of loops with value in the center. Note that any invariant inner product $\mathfrak{g}$ induces an inner product of $L\mathfrak{g}$ which restricts to an inner product on $L\mathfrak{z}$. This gives rise to a notion of projection and orthogonal complement.

Definition 1.8. Let Γ be the subgroup of LG of loops such that the velocity $(d\gamma)\gamma^{-1}$ has constant projection to $\mathfrak{z}$.

Definition 1.9. Let V be the orthogonal complement to $\mathfrak{z}$ in $L\mathfrak{z}$ with respect to the L^2 inner product. Then $\exp(V)$ is an abelian subgroup of LG.

In this definition, any subspace in V complementary to $\mathfrak{z}$ would suffice.

Proposition 1.10. *LG is the semidirect product of Γ and $\exp(V)$. If G is abelian, this is a direct product.*

Proof. The map $\Gamma \times \exp(V) \to LG$ of sets given by multiplication of loops is bijective. This follows from the fact that the map of Lie algebras $\mathfrak{L}_\Gamma \times V \to L\mathfrak{g}$ is a bijection, where $\mathfrak{L}_\Gamma$ denotes the Lie algebra of Γ.

The multiplication of $\Gamma \times \exp(V)$ such that the map is homomorphism of groups is given by $(\gamma_1, \exp(v_1))(\gamma_2, \exp(v_2)) = (\gamma_1\gamma_2, \exp(v_1)\exp(Ad(\gamma_1)v_2))$. If G is abelian, then Ad is always the identity, to the semidirect product becomes a product. □

If $G = T$, a torus, then the above proposition describes it as a product. Since $\mathfrak{z} = \mathfrak{t}$, we can identify Γ explicitly.

Lemma 1.11. *For a torus T, the space Γ of loops with velocity such that the projection $\mathfrak{z}$ is constant is isomorphic to $\Pi \times T$.*

Proof. LT can be described as the space of paths $\tilde{\gamma} : [0, 1] \to \mathfrak{t}$ such that $\tilde{\gamma}(1) - \tilde{\gamma}(0) \in \Pi$, modulo translation by Π. Then $d\gamma\gamma^{-1}$ is equal to the path $\tilde{\gamma}'$. The projection to $\mathfrak{z}$ is constant if and only if $\tilde{\gamma}'$ is constant. So Γ can be identified with the paths in $\mathfrak{t}$ modulo translation by Π which have constant velocity. The starting point of such a map is given by an element of $\mathfrak{t}/\Pi = T$. The velocity must be an element of Π otherwise $\tilde{\gamma}(1) - \tilde{\gamma}(0) \notin \Pi$. □

Now we use that for admissible representations of loops, $\exp(V)$ is a product of central extensions. Hence $(LT)^\tau \cong \Gamma^\tau \times (\exp(V))^\tau$. The $(\exp(V))^\tau$ has a unique irreducible unitary representation F, the Fock representation. So the positive energy representations of $(LT)^\tau$ are in bijective correspondence with the representations of Γ^τ. Hence we can apply the theory of the last section to obtain the following theorem:

Theorem 1.12. *There is an isomorphism of abelian groups*

$$\Psi : R^\tau(LG) \to K_T^{\tau + \dim T}(T)$$

It might seem weird that there is no contribution σ to the twist. But this twist is trivial over T and LT, since the adjoint representation maps everything to the identity in the case of abelian groups.

2 Dirac family construction for a compact group

2.1 Overview

The main source for this construction is [FHT07b, section 1], but there is also a summary in [FHT05, section 4]. An overview of the construction can be found in figure 2.1.

2.2 Spinor fields and a Dirac operator

2.2.1 The canonical Clifford extension of G

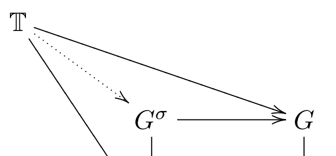
Let G be a compact Lie group. Fix a G-invariant inner product on $\mathfrak{g}$, i.e. $(-,-)$ on $\mathfrak{g}$ such that $(gX, gY) = (X, Y)$ for all $g \in G$. For a general compact Lie group averaging any inner product with respect to invariant measure gives us such an inner product. For semi-simple Lie groups, the Killing form suffices. Then the adjoint representation gives us a map $Ad : G \to O(\mathfrak{g})$.

Definition 2.1. The graded central extension G^σ is obtained as $G \times_{O(\mathfrak{g})} Pin^c(\mathfrak{g})$, with grading induced by the grading of $Pin^c(\mathfrak{g})$. We call it the canonical Clifford extension.

In the case of simple group, σ can be identified with the dual Coxeter number $\check{h}$, after identifying the twists with $\mathbb{Z}$.

Lemma 2.2. $1 \to \mathbb{T} \to G^\sigma \to G \to 1$ *is indeed a graded central extension and induces a split extension of Lie algebra* $0 \to i\mathbb{R} \to \mathfrak{g}^\sigma \to \mathfrak{g} \to 0$ *which has a splitting* $\mathfrak{g} \to \mathfrak{g}^\sigma$.

Proof. For the first statement, it suffices to prove that the kernel of the map $G^\sigma \to G$ is exactly $\mathbb{T}$. First note that the following diagram commutes, where the map $\mathbb{T} \to G$ is constant the identity and $\mathbb{T} \to Pin^c(\mathfrak{g})$ is the earlier inclusion, hence there is an induced map $\mathbb{T} \to G^\sigma$:



The kernel of $G^\sigma \to G$ is easy to describe if we use an explicit description of G^σ. G^σ is the subgroup of $G \times Pin^c(\mathfrak{g})$ of elements (g, x) such that $Ad(g) = T_x$. The maps $G^\sigma \to G$ and $G \to Pin^c(\mathfrak{g})$ are then simply the projections. This means that the kernel of $G^\sigma \to G$ consists of elements (e, x) such that $T_x = id$, where T_x is the element of $O(V)$ determined by a $x \in Pin^c$ (it is explicitly described in the appendix as $v \mapsto xv\alpha(x)^{-1}$). But $T_x = id$ if and only if x lies in the image of $\mathbb{T} \to Pin^c(\mathfrak{g})$. So we see that the kernel is exactly the image of $\mathbb{T} \to G^\sigma$.

The exact sequence of Lie algebra is a direct consequence of the first exact sequence. The splitting is induced by the splitting of $\mathfrak{pin}^c \to \mathfrak{o}$. ☐

2.2.2 The representation on spinor fields

Let S be the spinor representation of $Cl^c(\mathfrak{g}^*)$ with a compatible metric. This means that $Pin^c(\mathfrak{g})$ acts unitarily and therefore that G^σ acts unitarily on S through the map $G^\sigma \to Pin^c(\mathfrak{g})$. If $\dim G$ is odd there is a commuting $Cl^c(1)$ action. Furthermore, taking the infinitesimal representation of this unitary representation or equivalently restricting to $\mathfrak{pin}^c(\mathfrak{g}) \subset Cl^c(\mathfrak{g}^*)$, we get a Lie algebra representation of $\mathfrak{g}^\sigma$ on S.

Let $\{e_a\}$ be a basis of $\mathfrak{g}$ and $\{e^a\}$ be the dual basis of $\mathfrak{g}^*$. As in [FHT07b, 1.2], we define the following tensors using the inner product and Lie bracket on $\mathfrak{g}$:

$$\begin{aligned}
(e_a, e_b) &= g_{ab} \\
(e^a, e^b) &= g^{ab} \\
[e_a, e_b] &= f_{ab}^c e_c \\
([e_a, e_b], e_c) &= f_{abc}
\end{aligned}$$

Lemma 2.3. *The tensor f_{abc} is skew in all indices and satisfies $f_{ab}^c f_{cde} + f_{bd}^c f_{cae} + f_{da}^c f_{cbe} = 0$.*

Proof. The first statement is a consequence of the antisymmetry of the bracket: $[e_a, e_b] = -[e_b, e_a]$ and the invariance of the inner product $([e_a, e_b], e_c) = -(e_a, [e_b, e_c])$.

The second statement is a consequence of inserting the Jacobi identity in $(-, e_e)$. □

We will define spinor fields. There are three action on these spinor fields: the Clifford action γ of $\mathfrak{g}^*$, the spinor action σ of $\mathfrak{g}$ and the infinitesimal translation action R of $\mathfrak{g}$. We summarize the commutation relations for convenience and give details below:

$$[\gamma^a, \gamma^b] = -2g^{ab}$$

$$\begin{aligned}
[\sigma_a, \sigma_b] &= f_{ab}^c \sigma_c \\
[\sigma_a, \gamma^b] &= -f_{ac}^b \gamma^c \\
[R_a, R_b] &= f_{ab}^c R_c \\
[R_a, \gamma^b] &= [R_a, \sigma_a] = 0
\end{aligned}$$

Clifford action γ^a. We can let elements of $\mathfrak{g}^*$ act on S by Clifford multiplication. Denote the action of e^a by γ^a. Since S is a graded module, γ^a will necessarily be odd and since the metric is compatible, it will be a skew-Hermitian transformation. Being a Clifford representation, the graded commutator will be:

$$\boxed{[\gamma^a, \gamma^b] = -2g^{ab}}$$

Spinor action σ_a. We can let elements of $\mathfrak{g}$ act on S using the splitting $\mathfrak{g} \to \mathfrak{g}^\sigma$ and then mapping $\mathfrak{g}^\sigma$ into $\mathfrak{pin}^c$ to get a Lie-algebra reprsentation. By the following commutative diagram, it suffices to calculate the image in $\mathfrak{pin}^c$ of $ad(X) \in \mathfrak{so}(\mathfrak{g})$.

$$\begin{array}{ccc}
\mathfrak{g}^\sigma & \rightleftarrows & \mathfrak{g} \\
\downarrow & & \downarrow \\
\mathfrak{pin}^c & \rightleftarrows & \mathfrak{so}(\mathfrak{g})
\end{array}$$

The action of e_a can be calculated as follows. First calculate the components of $ad(e_a)$ with respect to the basis: these are given by $([e_a, e_b], e_c) = f_{abc}$. Hence in $\mathfrak{pin}^c$ we get the element $\sigma_a := \sum_{a<b} \frac{1}{4} f_{abc}(\gamma^b\gamma^c - \gamma^c\gamma^b) = \frac{1}{4} f_{abc}\gamma^b\gamma^c$ using skewness of f_{abc}. Note that σ_a will be even and since f_{abc} is skew and the γ^a are skew-Hermitian, it will be skew-Hermitian as well.

The fact that we are dealing with a Lie algebra representation gives us the graded commutator of σ_a and σ_b. However, one could also do this calculation directly using nothing but the formula for the commutator of γ^a's and the relation $f^c_{ab} f_{cde} + f^c_{bd} f_{cae} + f^c_{da} f_{cbe} = 0$. This gives us exactly the same result:

$$\boxed{[\sigma_a, \sigma_b] = f^c_{ab}\sigma_c}$$

So we have two actions of S. How do these two action interact?

Lemma 2.4. *The following formula holds:*

$$\boxed{[\sigma_a, \gamma^b] = -f^b_{ac}\gamma^c}$$

Proof. The proof uses the commutation relations for γ^a.

$$\begin{aligned}
[\sigma_a, \gamma^b] &= \frac{1}{4}(f_{acd}\gamma^c\gamma^d\gamma^b - f_{acd}\gamma^b\gamma^c\gamma^d) \\
&= \frac{1}{4}(f_{acd}\gamma^c\gamma^d\gamma^b + f_{acd}\gamma^c\gamma^b\gamma^d - 2f_{acd}g^{bc}\gamma^d) \\
&= \frac{1}{4}(f_{acd}\gamma^c\gamma^d\gamma^b - f_{acd}\gamma^c\gamma^d\gamma^b - 2f_{acd}g^{bc}\gamma^d + 2f_{acd}g^{db}\gamma^c) \\
&= \frac{1}{4}(-2f^b_{ac}\gamma^c - 2f_{adc}g^{db}\gamma^c) \\
&= \frac{1}{4}(-2f^b_{ac}\gamma^c - 2f^b_{ac}\gamma^c) = -f^b_{ac}\gamma^c
\end{aligned}$$

□

Alternatively, for a semisimple Lie algebra, one can simply define the action of $\mathfrak{g}$ using the σ commutation relations $[\sigma_a, \gamma^b] = -f^b_{ac}\gamma^c$ and $[\sigma_a, \sigma_b] = f^c_{ab}\sigma_c$ hold. These relations define σ_a, uniquely, for if σ'_a is a second such action, then $\sigma_a - \sigma'_a$ commutes with γ^b and since S is irreducible, must be a scalar λ_a. Then $f^c_{ab}\sigma_c = [\sigma_a, \sigma_b] = [\sigma_a + \lambda_a, \sigma_b + \lambda_b] = f^c_{ab}(\sigma_c + \lambda_c)$ implies that $f^c_{ab}\lambda_c = 0$. If $\mathfrak{g}$ is semisimple $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ implies that there exist complex numbers μ^a, υ^b such that $([\mu^a e_a, \upsilon^b e_b], e_d) = \delta^c_d$ and therefore $\mu^a \upsilon^b f^c_{ab} = \delta^c_d$ and we get $\lambda_c = 0$.

Differentation action R_a. Until now we have just considered spinors, not spinor fields. Because $\mathfrak{g}$ can be identified with left-invariant vector fields of G, we get a Lie algebra action of $\mathfrak{g}$ on $C^\infty(G)$ by applying the corresponding vector fields to functions. Denote the action of e_a by R_a. This satisfies:

$$\boxed{[R_a, R_b] = f^c_{ab}R_c}$$

Now consider $C^\infty(G) \otimes S$, which we can identify by left translation with $C^\infty(G, S)$, the spinor fields. In this situation R_a are even self-adjoint operators. Because we trivialized using left translation and the R_a are left-invariant vector fields, we get the following interaction between the R_a and γ^a and σ_a:

$$\boxed{[R_a, \gamma^b] = [R_a, \sigma_a] = 0}$$

The space $C^\infty(G) \otimes S$ embeds into $L^2(G) \otimes S$, where the L^2 norm is with respect to the invariant measure. $L^2(G)$ with left translation by G is a very useful representation, playing the role of fundamental representation for compact Lie groups. This means we have the following decomposition theorem.

Theorem 2.5 (Corollary of Peter-Weyl)**.** *$L^2(G) \otimes S$ decomposes as follows:*

$$L^2(G) \otimes S \cong \bigoplus_{V \in \hat{G}} V^* \otimes V \otimes S$$

where $\bigoplus$ denotes the completed tensor product, $\hat{G}$ the set of irreducible representations of G, G acts on V^* by right translation, on V by left translation and projective on S through G^σ.

The identification of V with a subspace of $L^2(G)$ is done using matrix coefficients, which are smooth functions. Therefore each summand is a finite dimensional space is smooth spinor fields and the actions of $\mathfrak{g}$ and $\mathfrak{g}^*$ described earlier restrict to actions on the summand. Furthermore, which these actions only work on S or on V by left translation, the action on V^* is trivial and we can forget about it. From now on, we will be interested in $V \otimes S$ only.

2.2.3 Konstant's cubic Dirac operator

The skewness of f_{abc} allows us to define the following element of $\Lambda^3(\mathfrak{g}^*)$:

$$\Omega = \frac{1}{6} f_{abc} e_a \wedge e_b \wedge e_c$$

It is G-invariant for the action induced by the coadjoint action on $\mathfrak{g}^*$ as a consequence of the invariance of $(-,-)$. This allows us to introduce the following Dirac operator on $C^\infty(G) \otimes S$:

Definition 2.6 (The Dirac operator D_0)**.** Define $D_0 : C^\infty(G) \otimes S \to C^\infty(G) \otimes S$ as follows:

$$D_0 = i\gamma^a R_a + \frac{i}{3}\gamma^a \sigma_a = i\gamma^a R_a + \frac{i}{12} f_{abc}\gamma^a\gamma^b\gamma^c = i\gamma^a R_a + \frac{i}{2}\gamma(\Omega)$$

where $\gamma(\Omega)$ is given by the analogue $\Lambda^3(\mathfrak{g}^*) \to Cl^c(\mathfrak{g}^*)$ of the canonical map $\Lambda^2(\mathfrak{g}) \to Cl^c(\mathfrak{g}^*)$ defined earlier.

One can took a look at Landweber's article for applications of this Dirac operator to an analogue of the Borel-Bott-Weil theorem for loop groups.

Proposition 2.7. *D_0 is an odd skew-adjoint operator. It is G^σ-invariant. The square is given as follows:*

$$D_0^2 = g^{ab}(R_a R_b + \frac{1}{3}\sigma_a \sigma_b)$$

so we can conclude that D_0 is indeed a Dirac operator (its principal symbol is that of a generalized Laplacian).

Proof. The first statement is trivial consequence of the definition. For G^σ-invariance, remember that G^σ acts on $C^\infty(G)$ through G by left translation and on S through $Pin^c(\mathfrak{g})$. Let $(g, x) \in G^\sigma$, A_a^b given by $Ad(g)(e_a) = A_a^b e_b$. Note that the the coadjoint action on $e^a = (e_a, -)$ is given by the contragredient action $g(e_a, -) = (e_a, g^{-1}-)$ and hence $CoAd(g)(e^a) = (A^{-1})_b^a e^b$. From this we conclude that $(g, x)R_a(f)(h) = A_a^b R_b(f)(gh)$, $(g, x)\sigma_a(\psi) = A_a^b \sigma_b(x\psi)$ for $\psi \in S$ and $(g, x)\gamma^a(\psi) = (A^{-1})_b^a \gamma^b(x\psi)$. Using this, we establish G^σ-invariance:

$$\begin{aligned}
(g, x)D_0(f \otimes \psi(h)) &= (g, x)iR_a(f) \otimes \gamma^a\psi(h) + (g, x)f \otimes \frac{i}{3}\gamma^a\sigma_a\psi(h) \\
&= iA_a^b R_b(f) \otimes (A^{-1})_c^a \gamma^c(x\psi)(gh) + f \otimes \frac{i}{3}(A^{-1})_c^a \gamma^c A_a^b \sigma_b(x\psi)(h) \\
&= D_0(f \otimes (x\psi)(gh)) = D_0((x, g)f \otimes \psi)(h)
\end{aligned}$$

Note that we could have used the invariance of $\gamma\Omega$ to replace a part of the above calculation, but two variations of a proof are better than one. To compute the square, we note that since D_0 is odd, $D_0^2 = \frac{1}{2}[D_0, D_0]$. Now it is just a matter of applying the commutation relations. □

Note that the invariance and the fact that only left translation appears, imply D_0 restricts to $V \otimes S$.

2.3 The family of cubic Dirac operators on an irreducible representation

2.3.1 The definition and first properties

Our next goal is to define a family of Dirac operators in $End(V \otimes S)$ depending on a paramter in $\mathfrak{g}^*$. We do this in the simplest way possible, using $\mathfrak{g}^*$ to define an additional Clifford multiplication.

Definition 2.8. Define $D(V) : \mathfrak{g}^* \to End(V \otimes S)$ as follows:

$$\mu = \mu_a e^a \mapsto D_\mu = D_0 + \mu_a \gamma^a = D_0 + \gamma(\mu)$$

Proposition 2.9. *For each $\mu \in \mathfrak{g}^*$, D_μ is an odd skew-adjoint operator, i.e. D_μ restricts as $D_\mu : V \otimes S^+ \to V \otimes S^-$ and $D_\mu : V \otimes S^- \to V \otimes S^+$, and G^σ-equivariant, i.e. $(g,x)D_\mu = D_{CoAd(g)(\mu)}(g,x)$. Its square is given as follows:*

$$D_\mu^2 = D_0^2 - |\mu|^2 - 2i\mu_b g^{ba}(R_a + \sigma_a)$$

so we can conclude that D_0 is indeed a Dirac operator (its principal symbol is that of a generalized Laplacian).

Proof. The first statement is trivial. The second follows from G^σ-invariance of D_0 combined with the fact that $(g,x)\mu_a\gamma^a = \mu_a(A^{-1})^a_b\gamma^b = CoAd(\mu)_b\gamma^b$.

To calculate the square, we note that since D_μ is odd, we have $D_\mu^2 = \frac{1}{2}[D_\mu, D_\mu]$. This can be expanded as $D_\mu^2 = D_0^2 + \frac{1}{2}[\mu_a\gamma^a, \mu_a\gamma^a] + [D_0, \mu_a\gamma^a]$, where we use the fact that the graded commutator of two odd elements in symmetric in this entries. Using the commutating relations, we derive $[\mu_a\gamma^a, \mu_b\gamma^b] = -2\mu_a\mu_b g^{ab} = -2|\mu|^2$ and

$$\begin{aligned}[D_0, \mu_a\gamma^a] &= [i\gamma^a, \mu_b\gamma^b]R_a + \frac{i}{3}[\gamma^a\, \mu_d\gamma^d] \\ &= -2i\mu_b g^{ab}R_a - 2i\mu_b g^{ab}\sigma_a\end{aligned}$$

This suffices to prove the proposition. □

2.3.2 The kernel

We want to know over which points of $\mathfrak{g}^*$ the family $D(V)$ induces an isomorphism of the fiber and over which it doesn't. Since the fibers are finite dimensional, it suffices to find to kernel of D_μ. We start by doing the calculation in the case that G is connected. After that we'll formulate a proposition for general G.

To do the calculation, we decompose our irreducible representation V in a convenient way. Fix a $\mu \in \mathfrak{g}^*$ and fix a maximal torus T_μ in $Z_\mu \subset G$. We have chosen this for the following reason: it holds that for $t \in T_\mu$, $CoAd(t)(\mu)(X) = \mu(Ad(t^{-1})(X)) = \mu(X)$. Infinitesimally, this implies $\mu([H,X]) = 0$ for $H \in \mathfrak{t}_\mu$. Since $\mu([H,X]) = i\alpha(H)\mu(X)$ for $X \in \mathfrak{g}_\alpha$, we conclude that μ now annihilates all non-zero root spaces. This is equivalent to $\mu \in \mathfrak{t}_\mu^*$.

If μ is regular, then we can choose a Weyl chamber such that μ is antidominant. If μ is not regular, we can chosen a Weyl chamber such that μ is a negative wall of the dual Weyl chamber. After fixing a Weyl chamber, we can decompose $\mathfrak{g}_\mathbb{C}$ as $\mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha\in\Delta^+}(\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha)$, where $\Delta^+ \subset \mathfrak{t}^*$ are the positive roots. Define $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha \in \mathfrak{t}_\mu^*$. Note that it depends on μ.

We give an alternative proof to the one in [FHT07b, proposition 1.19]. This is the one suggested in [FHT07b, footnote 7] and [FHT05, section 4.1].

Proposition 2.10. *Suppose V is irreducible with lowest weight $-\lambda$. Then $D_\mu \in End(V \otimes S)$ is nonsingular unless μ is regular and $\mu = -\lambda - \rho$. In the latter case* $\ker D_\mu = K_{-\lambda} \otimes S_{-\rho}$*, where $K_{-\lambda} \subset V$ is the one-dimensional root space of lowest weight $-\lambda$ and $S_{-\rho} \subset S$ is the root space of lowest weight $-\rho$.*

Proof. The proof proceeds in the following steps:

(a) Show D_0^2 is multiplication by a real constant and D_0 is Clifford multiplication on $K_{-\lambda} \otimes S_{-\rho}$.

(b) Link D_μ^2 to D_0^2.

(c) Show that D_μ^2 has value 0 on $K_{-\lambda} \otimes S_{-\rho}$ and a lower value on all other weight spaces.

(d) Link D_μ to D_0 on $K_{-\lambda} \otimes S_{-\rho}$ and show that $\ker D_\mu$ coincides with $\ker D_\mu^2$.

Let's proceed with this program.

(a) Let $\dot{\pi} : \mathfrak{g} \to End(V \otimes S)$ be the total infinitesimal action on $V \otimes S$. It is given by

$$\dot{\pi}(\xi) = \xi^a (R_a + \sigma_a)$$

if $\xi = \xi^a e_a \in \mathfrak{g}$. Then it is easy to check that the following two commutation relations hold:

$$[D_0^2, \dot{\pi}(\xi)] = 0$$
$$[D_0^2, \gamma(\xi^a g_{ab} e^b)] = 0$$

Because V is irreducible V is generated from the lowest weight vector by applying $\gamma(\xi^a g_{ab} e^b)$ and $\dot{\pi}(\xi)$. This means that D_0^2 is determined by its value on the lowest weight space of V, which is $K_{-\lambda} \otimes S_{-\rho}$.

We will show that D_0^2 is multiplication by a constant on $K_{-\lambda} \otimes S_{-\rho}$. To do this, write $\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha \in \Delta^+} (\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha)$ and choose a compatible basis e_{t_j}, e_α, $e_{-\alpha}$ (it is well-known that the $\mathfrak{g}_\alpha$ are one-dimensional).

For $\alpha \in \Delta^+$, we rewrite the term $\frac{i}{3}\gamma^\alpha \sigma_\alpha$ as $\frac{i}{3}\sigma_\alpha \gamma^\alpha + \frac{i}{3} f^\alpha_{\alpha c} \gamma^c$ using the commutation relations. Note that $f^\alpha_{\alpha c}$ is non-zero only if $c = t_j$ and then has value $-i\alpha(e_{t_j})$, hence $\frac{i}{3} f^\alpha_{\alpha c} \gamma^c = \frac{1}{3}\gamma(\alpha)$. Summing over $\alpha \in \Delta^+$, this will contribute a term $\frac{2}{3}\gamma(\rho)$. Now we can write D_0 as follows:

$$D_0 = i(\gamma^\alpha R_\alpha + \gamma^{-\alpha} R_{-\alpha} + \gamma^{t_j} R_{t_j}) + (\frac{i}{3}\gamma^{-\alpha}\sigma_{-\alpha} + \frac{i}{3}\sigma_\alpha \gamma^\alpha + \frac{i}{3}\gamma^{t_j}\sigma_{t_j}) + \frac{2}{3}\gamma(\rho)$$

Note that since $K_{-\lambda} \otimes S_{-\rho}$ is of lowest weight, $R_{-\alpha}$ and $\sigma_{-\alpha}$ for $\alpha \in \Delta^+$ will vanish on it. Furthermore, since $[\sigma_t, \gamma^\alpha] = -f^\alpha_{tc}\gamma^c$ and $f^\alpha_{t_j c}$ is non-zero only if $c = \alpha$ and then has value $i\alpha(e_{t_j})$. This shows that γ^α maps the weight space of S of weight ω to that of weight $\omega - \alpha$. Therefore γ^α will vanish on the lowest weight space as well.

These considerations imply that the terms $\gamma^\alpha R_\alpha$ and $\gamma^{-\alpha} R_{-\alpha}$ vanish and only the terms coming from $\mathfrak{t}_\mathbb{C}$ survive. Similarly, the terms $\gamma^{-\alpha}\sigma_{-\alpha}$ and $\sigma_\alpha\gamma^\alpha$ vanish and only the terms coming from $\mathfrak{t}_\mathbb{C}$ survive. So on $K_{-\lambda} \otimes S_{-\rho}$ we can write D_0 as:

$$D_0 = i\gamma^{t_j} R_{t_j} + \frac{i}{3}\gamma^{t_j}\sigma_{t_j} + \frac{2}{3}\gamma(\rho)$$

Now note that R_{t_j} acts as multiplication by $-i\lambda(e_{t_j})$ and σ_{t_j} acts as multiplication by $-i\rho(e_{t_j})$. There we obtain that

$$D_0 = \gamma(\lambda) + \frac{1}{3}\gamma(\rho) + \frac{2}{3}\gamma(\rho) = \gamma(\lambda + \rho)$$

It is now a trivial consequence of the relations in the Clifford algebra that

$$D_0^2 = -|\lambda + \rho|^2$$

(b) We introduce two operators adapted to μ. The first is the μ-shifted $\mathfrak{g}$-action $\dot{\pi}_\mu : \mathfrak{g} \to End(V \otimes S)$. It is given by:

$$\dot{\pi}_\mu(\xi) = \xi^a(R_a + \sigma_a - i\mu_a)$$

if $\xi = \xi^a e_a \in \mathfrak{g}$. It is skew-adjoint.

The second is the μ-shifted energy $E_\mu \in End(V \otimes S)$. It is given by:

$$E_\mu = i\dot{\pi}_\mu(g^{ab}\mu_a e_b) - \frac{|\mu|^2}{2} = i\mu_a g^{ab}(R_b + \sigma_b) + \frac{|\mu|^2}{2}$$

It is self-adjoint. We can claim that E_μ has constant value $(\frac{\mu}{2} - \omega, \mu)$ on the weight space of weight ω. This is a simple consequence of the definition of the μ-shifted action. On the weight space of weight ω, $\dot{\pi}_\mu$ equals multiplication by $i(\mu, \omega)$. Then E_μ is easily seen to be multiplication by $-(\mu, \omega) + \frac{|\mu|^2}{2} = (\frac{\mu}{2} - \omega, \mu)$.

Our earlier calculations in proposition 2.9 show that $D_\mu^2 + 2E_\mu = D_0^2$. This implies that $D_\mu^2 + 2E_\mu$ is multiplication by the constant $-|\lambda + \rho|^2$, which is therefore independent of μ.

(c) We look a bit closer at E_μ. Note that the antidominancy of μ implies that $-(\mu, \omega)$ and therefore $(\frac{\mu}{2} - \omega, \omega)$ is minimal on the lowest weight. Note that if μ is regular, it has strictly higher value on other weights. Otherwise μ might have the same value on some other weights.

Anyway, since E_μ and D_0^2 are multiplication by a constant on weight spaces, D_μ^2 is as well and we can conclude that the value D_μ^2 is maximal on on the weight space $K_{-\lambda} \otimes S_{-\rho}$. To get this maximum, we calculate the value D_μ^2 on $K_{-\lambda} \otimes S_{-\rho}$ using the value of D_0^2 and E_μ there.

Note that on $K_{-\lambda} \otimes S_{-\rho}$ the operator $2E_\mu$ is multiplication by $|\mu|^2 + 2(\rho + \lambda, \mu)$. This implies that D_μ^2 is multiplication by

$$-|\lambda + \rho|^2 - |\mu|^2 - 2(\rho + \lambda, \mu) = -|\lambda + \rho + \mu|^2$$

From this we get $D_\mu^2 = 0$ on $K_{-\lambda} \otimes S_{-\rho}$ if and only if $-|\mu + \lambda + \rho|^2 = 0$ if and only if $\mu = -\lambda - \rho$. Furthermore, because D_μ^2 is nonpositive, if D_μ^2 is not zero on $K_{-\lambda} \otimes S_{-\rho}$, it is not nonsingular on the entire space $V \otimes S$.

Note that if μ is not regular, then the fact that μ lies on the wall of some dual Weyl chamber implies that $\mu + \rho \notin \mathbb{Z}\Delta^+$. But $\lambda \in \mathbb{Z}\Delta^+$, being a root, hence $-|\mu + \lambda + \rho|^2 \neq 0$.

(d) Finally, to check that $\ker D_\mu^2 = \ker D_\mu$ we note that our calculation that D_0 is $\gamma(\lambda + \rho)$ on $K_{-\lambda} \otimes S_{-\rho}$ implies that D_μ is equal to $\gamma(\mu + \lambda + \rho)$ there. We conclude that for $\psi \in K_{-\lambda} \otimes S_{-\rho}$, $|\gamma(\mu + \lambda + \rho)\psi|^2 = |\mu + \lambda + \rho|^2 |\psi|^2$, using the skew-adjointness of the γ^a, and therefore $\gamma(\mu + \lambda + \rho)\psi = 0$ if and only if $|\mu + \lambda + \rho|^2 = 0$.

Alternatively, one can use that D_μ is a skew-adjoint operator and therefore if $\psi \in \ker D_\mu^2$, $(D_\mu \psi, D_\mu \psi) = (D_\mu^2 \psi, \psi)$ implies that $\psi \in \ker D_\mu$ as well.

□

We made the assumption that G is connected. If this is not the case, there are some minor changes. $K_{-\lambda}$ doesn't have to be one-dimensional any more, but Z_μ acts irreducibly on $K_{-\lambda}$. This means that D_0^2 must still act as a constant on $K_{-\lambda}$ and the theorem continues to hold.

2.4 Associated K-classes

2.4.1 The main theorem for compact groups

Note that each D_μ is a Fredholm operator, since it acts on a finite-dimensional space. Therefore we can consider $D(V)$ as a family of odd skew-adjoint $\mathbb{Z}/2\mathbb{Z}$-graded Fredholm operator. This family is compactly supported, i.e. invertible outside a compact subset, and G^σ-equivariant. If $\dim G$ is odd, there is a commuting $Cl^c(1)$-action. Therefore it corresponds to a class as follows

$$[D(V)] \in K_G^{\sigma+\dim G}(\mathfrak{g}^*)_{cpt}$$

where σ is an abbreviation for the twisting $(\mathfrak{g}^* \to \mathfrak{g}^*, G^\sigma, \epsilon^\sigma)$. These maps assemble into a map $\Psi : R(G) \to K_G^{\sigma+\dim G}(\mathfrak{g}^*)_{cpt}$. Then the following theorem holds:

Theorem 2.11. *Let G be a compact Lie group. Then there is an isomorphism of abelian groups:*

$$\boxed{\Psi : R(G) \to K_G^{\sigma+\dim G}(\mathfrak{g}^*)_{cpt}}$$

To prove this we will require the standard isomorphism $K_G^0(pt) \cong R(G)$.

Proof. The Thom isomorphism for the G-bundle $\mathfrak{g}^*$ over a point in twisted equivariant K-theory is usually given by a pushforward of $j : \{0\} \hookrightarrow \mathfrak{g}^*$ (see [FHT07a, section 3.6]). It assigns to a representation the family of Clifford multiplication operators, supported at the origin.

$$\begin{aligned} j : K_G^0 &\to K_G^{\sigma+\dim G}(\mathfrak{g}^*)_{cpt} \\ V &\mapsto (V \otimes S, 1 \otimes \gamma) =: [\gamma] \end{aligned}$$

Let $\pi : \mathfrak{g}^* \to \{0\}$ be the projection. Consider the following diagram:

$$K_G^0 \underset{\pi_*}{\overset{j_*}{\rightleftarrows}} K_G^{\sigma+\dim G}(\mathfrak{g}^*)_{cpt}$$

By naturality j_* and π_* are inverse. It suffices to show that $j_*([V]) = [D(V)]$. For this note that γ and $D(V)$ through compactly supported families of operators $\epsilon D_0 + \mu_a \gamma^a$, with $\epsilon \in [0,1]$, hence $D(V)$ is a compact perturbation of γ. This implies $[\gamma] = [D(V)]$. □

2.4.2 Twisted K-theory of coadjoint orbits

The kernels patch together to a G^σ-equivariant bundle $K \otimes S'$ over the coadjoint orbit $\mathcal{O}$ of $\mu = -\lambda - \rho$. The subbundle K is G-equivariant and S' is a twisted bundle of twist σ. The spinor bundle of the normal bundle to $\mathcal{O}$ has twists $\sigma(N)$. As a consequence of the bundle $L = Hom_{Cl^c(N)}(S(N), S')$ has twist $\sigma - \sigma(N)$. But $T\mathcal{O} \oplus N$ is trivial, hence the corresponding spinor bundles $S(\mathcal{O})$ gives a twisting $\sigma(\mathcal{O}) = \sigma - \sigma(N)$. So $K \otimes L$ is a $\sigma(\mathcal{O})$-twisted bundle. If G is connected and simply-connected, $\sigma(\mathcal{O}) = \sigma(\mathcal{N}) = \sigma = 0$, since G admits no nontrivial twistings.

We then obtain the following commutative diagram:

$$\begin{array}{ccc} & K_G^{\sigma(\mathcal{O})+\dim \mathcal{O}}(\mathcal{O}) & \\ {}^{p_*}\swarrow & & \searrow^{k_*} \\ K_G^0(pt) & \underset{\pi_*}{\overset{j_*}{\rightleftarrows}} & K_G^{\sigma+\dim G}(\mathfrak{g}^*)_{cpt} \end{array}$$

We already know that $\pi_* = (j_*)^{-1}$ and by construction $k_*([K \otimes L]) = [D(V)]$ and $p_*([K \otimes L]) = V$ as a consequence.

2.5 Examples

2.5.1 Finite groups

In the case the Lie algebra, Dirac operator, spinors and the associated twisted σ are zero. The theorem then becomes trivial, the isomorphism going between two equal groups.

2.5.2 Tori

We compare the case of tori to our earlier construction of K-classes for tori. Note that $f_{abc} = 0$, since T is abelian, and therefore the Dirac operator reduces to the one we considered early on $L\mathfrak{t}$. Alternatively, one can construct that Dirac operators used there by pullback of the one in the compact group case.

On the one hand we have the representation theory, which says $R(G) = \mathbb{Z}(\Pi^*)$, where $\Pi^* = Hom(T, \mathbb{T}) = Hom(\Pi, \mathbb{Z})$ are the characters. All these representations are one-dimensional, so $D_\mu : End(V_\lambda \otimes S)$ for $\lambda \in \Pi^*$ reduces to $v \otimes \xi \mapsto i\lambda_a \gamma^a + i\mu_a \gamma^a$, where we identify λ with a linear map of $\mathfrak{t}$ which sends Π to the integers. Therefore the kernel of D_μ is exactly in supported in $\{-\lambda\} \subset \mathfrak{t}^*$. Since T is abelian, the coadjoint orbits consists of points.

In the special case $T = S^1$ we can be even more explicit. We have $\mathfrak{t} = i\mathbb{R}$, which is generated by i of norm 1 if we take the inner product $(ia, ia') = aa'$. Then we identify $\mathfrak{t}^*$ with $i\mathbb{R}$ through this, and the dual vector to i is i. Under these choices, if we fix $i\mathbb{R}_{\geq 0}$ as the positive Weyl chamber, we have that $\Pi^* = i\mathbb{Z}$ and $\rho = 0$, since no roots appear in $\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha \in \Delta^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$. The spinor representation S is given by $\mathbb{C} \oplus \mathbb{C}$, where γ^t is given by:

$$\gamma^i = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Each irreducible is labelled by a single weight in. Then D_{ia} on $V_{in} \otimes S$ is given by:

$$D_\mu = \begin{pmatrix} 0 & i(n+a) \\ i(n+a) & 0 \end{pmatrix}$$

Indeed, $D_{ia} = \gamma(in + ia)$ on the lowest weight space and $D_{ia}^2 = -(n+a)^2 Id$. Furthermore D_{ia} is singular only if $ia = -in$, so everything works out fine.

On the other hand, we have the twisted K-theory. First note that σ will be trivial over T, because $Ad : T \to O(\mathfrak{t})$ is the constant map with value identity. Therefore, we are actually dealing with the usual equivariant K-theory. Each $[D(V)]$ is a class in $K_T^{\dim T}(\mathfrak{t}^*)_{cpt}$ supported above its lowest weight character, which is easily seen to be isomorphic to $K_T^0(pt) = R(T)$. Alternatively, one can see this as an instance of the usual Thom isomorphism in equivariant K-theory.

3 Dirac family construction for loop groups

3.1 Overview

The main source for this construction is [FHT07b, sections 2-5], but there is also a summary in [FHT05, section 8-13] for more general cases. An overview of the construction can be found in figure 3.1. There is a proof for the case that G is a finite group in [Wil05].

3.2 Compact groups and loop groups: differences and similarities

Next we turn to the study of loop groups. Our goal will be to generalize the results of the last paragraphs to loop groups. That is, we want to find a map from some variation on the representation ring to some twisted K-theory group and show that this is an isomorphism.

Of course, since the authors of the papers [FHT07a], [FHT07b] and [FHT05] thought about the way they set up their articles, the general idea of the procedure remains the same. Again we will define a spinor representation, take a look at spinor fields, define families of Dirac operators for representations and use these to give a map. However, technically things become more involved:

- We will be interested in representations of central extensions of loop groups, because all interesting representations are projective. Additional complications will come from the fact that not every central extension support enough structure to make the theory work. Therefore we restrict ourselves to admissible extensions (don't worry, for semisimple compact Lie groups, every central extension is admissible [FHT07b, proposition 2.15]).

More generally, we can extend our theory to gauge transformations of principal G-bundles over the circle. These are called twisted loop groups by Pressley-Segal [Pre86, section 3.7].

- The only representations for which a theory similar to that of compact Lie groups works are the so-called positive energy representations. However, these representations are in general infinite-dimensional. Furthermore, the Lie algebra is infinite-dimensional. All this makes the definitions harder and requires us to use functional analysis in some places.
- The Dirac family will no longer be indexed by $\mathfrak{g}^*$, but by the affine space $\mathcal{A}_P$ of connections of the principal G-bundle over the circle. These will be isomorphic with the affine space of linear splittings of $(L_P\mathfrak{g})^\tau \to L_P\mathfrak{g}$, as in the case of projective representations of compact Lie groups [FHT07b, section 1.5].
- There will be some convergence problems. To deal with these we need to work on the dense subspace of finite energy loops. For this we need a correct notion of energy with respect to a connection.

The result will be the following theorem. in its most general form:

Theorem 3.1. *Let $(L_PG)^\tau$ be a positive definite admissible graded central extension of L_PG. Then there is an isomorphism of graded free abelian groups*

$$\boxed{\Psi : R^{\tau-\sigma}(L_PG) \to K_G^{\tau+\dim G}(G[P])}$$

where $R^{\tau-\sigma}(L_PG)$ denotes the free abelian group on the irreducible positive energy representations at level $\tau - \sigma$ and $K_G^{\tau+\dim G}(G[P])$ denotes the twisted equivariant K-theory of the image of the holonomy of P in G.

3.3 Sketch of construction

We will give a sketch in the untwisted case, i.e. $P = G \times S^1$, and provide some additional details in each section, mainly for my own benefit.

3.3.1 L_PG and admissible central extensions

Sketch. A loop group LG is the space of smooth loops in a Lie group. This has a Lie algebra $L\mathfrak{g}$. Because we will study projective representations, we want to look at central extensions of LG by $\mathbb{T}$. To get a good definition of energy, we need to able to rotate loops and therefore look at the slightly larger group $\hat{L}G$. Admissible extensions are those central extensions with the correct properties to define the energy with respect to a connection and the Dirac operator in later sections: there is a compatibility with the central extension and $\hat{L}G$. Admissible extensions occur often and have nice properties.

Details. We start with the definition of a loop groups and its extension to free rotation loops.

Definition 3.2. For G a compact Lie group and LG denote the space of smooth loops in G. Note that this is the same as the space of G-equivariant diffeomorphisms of $G \times S^1$ convering the identity of S^1. Furthermore, let $\hat{L}G$ denote the group of G-equivariant diffeomorphisms of $G \times S^1$ convering a rigid rotation S^1.

The topology and manifold structure on LG are considered in [Pre86, section 3.2]. We note that the group $\hat{L}G$ can be described as a semidirect product of LG with S^1. The product of $(\gamma, \varphi)(\eta, \phi)$ is given by $(r_\phi(\gamma)\eta, \varphi + \phi)$, where $(r_\phi(\gamma)\eta(\varphi))(\theta) = \gamma(\theta + \phi)\eta(\theta)$. The group $\hat{L}G$ is very useful to consider, because it gives us elements gives rotate loops. For this reason, it also occurs naturally in the theory of the root system and Kac-Moody algebras, see e.g. [Pre86, chapter 5] and in particular [Pre86, page 71].

The definition of $\hat{L}G$ gives short exact sequences of groups and Lie algebras:

$$1 \to LG \to \hat{L}G \to \hat{\mathbb{T}}_{rot} \to 1$$

$$1 \to L\mathfrak{g} \to \hat{L}\mathfrak{g} \to i\hat{\mathbb{R}}_{rot} \to 1$$

Because we want to deal with projective representations, we need to look at central extensions of LG by $\mathbb{T}$. However, we need to restrict to a special class of central extensions, the admissible ones. They roughly have the properties that the central extension can be extended to $\hat{L}G$ and that we have a nice inner product on the Lie algebra of this extension $(\hat{L}G)^\tau$.

Definition 3.3. A central extension $(LG)^\tau$ is admissible if:

(a) There is a compatible extension $(\hat{L}G)^\tau$ of $\hat{L}G$, which means that the following diagram is commutative:

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{T} & \longrightarrow & (LG)^\tau & \longrightarrow & LG & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{T} & \longrightarrow & (\hat{L}G)^\tau & \longrightarrow & \hat{L}G & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{T}_{rot} & = & \mathbb{T}_{rot} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

In this case, there is a commutative diagram of Lie algebras as follows:

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & i\mathbb{R} & \longrightarrow & (L\mathfrak{g})^\tau & \longrightarrow & L\mathfrak{g} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & i\mathbb{R} & \longrightarrow & (\hat{L}\mathfrak{g})^\tau & \longrightarrow & \hat{L}\mathfrak{g} & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & i\mathbb{R}_{rot} & = & i\mathbb{R}_{rot} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

Let K denote the generator $i \in i\mathbb{R}$.

(b) On $(\hat{L}\mathfrak{g})^\tau$ there exists a $(\hat{L}G)^\tau$-invariant inner product $\ll -, - \gg$ such that $\ll K, d \gg = -1$ for all $d \in (\hat{L}\mathfrak{g})^\tau$ which are mapped to $i \in i\mathbb{R}_{rot}$.

This extension and inner product are considered part of the data of an admissible central extension.

The class of admissible extensions can be enlarged to that of topologically and analytically regular extensions [FHT05, section 2]. Note that the d's in the definition of admissible extension are exactly G-invariant vector fields tranverse to the fibers, thus generating a rotation-like flow. We first note some trivial and non-trivial properties of the class of admissible extensions:

Proposition 3.4. ***(a)*** *Let G be a semisimple Lie group, then every central extension of LG is admissible and for each $(\hat{L}G)^\tau$ the inner product is unique.*

(b) *If G is simply connected, then the isomorphism classes of central extensions are in bijection with elements of $H^2(LG;\mathbb{Z})$.*

(c) *Let $(LG)^{\tau_1}$ and $(LG)^{\tau_2}$ be two admissible central extensions. Then $(LG)^{\tau_1+\tau_2} := (LG)^{\tau_1} \times_{LG} (LG)^{\tau_2}$ is also admissible.*

(d) *For every admissible central extension $(LG)^\tau$ there exists an inverse admissible central extension $(LG)^{-\tau}$.*

Sketch of proof of part (a). This proof is an elaborate version of Pressley-Segal's proof that every central extension of a simply connected G is determined by a G-invariant bilinear form on $\mathfrak{g}$, using the density of polynomial loops. □

The importance of the inner product is the following proposition, linking it to connections and splittings of the map $(\hat{L}\mathfrak{g})^\tau \to (L\mathfrak{g})^\tau$ (compare with the case of projective representations [FHT07b, section 1.5]). First we need a lemma as preparation:

Lemma 3.5. *The inner product $\ll -,- \gg$ has the following properties:*

$$\ll K, (LG)^\tau \gg = 0$$

$$\ll K, K \gg = 0$$

Proof. Note that the second statement is a consequence of the first. The first follows by noting that if we fix a $d \in (\hat{L}\mathfrak{g})^\tau$ which is mapped to $i \in i\mathbb{R}_{rot}$, then for all $\xi \in (LG)^\tau$ there exists a unique $d' \in (\hat{L}_P\mathfrak{g})^\tau$ which is mapped to $i \in i\mathbb{R}_{rot}$ such that $d - d' = \xi$, where we identify $(LG)^\tau$ with its image in $(\hat{L}\mathfrak{g})^\tau$. This implies that $\ll K, \xi \gg = \ll K, d \gg - \ll K, d' \gg = -1 + 1 = 0$. □

Let's remind ourselves a bit about connections in principal G-bundles. There are many equivalent definitions, most importantly as horizontal distributions, collections of lifts of vector fields of the base space or simply as an affine space $\mathcal{A}$ (a $\Omega^1(S^1;\mathfrak{g}) \cong L\mathfrak{g}$-torsor). For the untwisted case, there is a distinguished connection, corresponding to the canonical horizontal distribution A_0 of the product $G \times S^1$. The correspoding G-invariant horizontal vector field d_{A_0} is $\frac{d}{d\theta}$. Using this canonical connection, the $L\mathfrak{g}$-torsor $\mathcal{A}_p$ can be identified with $\Omega^1(S^1, \mathfrak{g})$, the $\mathfrak{g}$ valued one-forms.

Proposition 3.6. *Let $(LG)^\tau$ be an admissible extension. If $\ll -,- \gg$ is nondegenerate, then the following sets of data are equivalent:*

(a) *A connection $A \in \mathcal{A}$.*

(b) *A lift of the horizontal vertical field $d_A \in \hat{L}\mathfrak{g}$ to an element d_A of $(\hat{L}\mathfrak{g})^\tau$ such that $\ll d_A, d_A \gg = 0$.*

(c) *A $(L\mathfrak{g})^\tau$-equivariant splitting of the map $(\hat{L}\mathfrak{g})^\tau \to (L\mathfrak{g})^\tau$.*

If $\ll -,- \gg$ is not nondegenerate, then (a) and (b) are equivalent, and a datum of type (b) gives one of type (c). However, the latter map might not be a bijection.

Proof. **(a) ⇔ (b)** Any connection is completely determined by its horizontal distribution. The vector field d_A, the unique horizontal lift of the vector field $\frac{d}{d\theta}$ on the circle, thus determines the connection uniquely. Now we must show that for each inner product and each d_A, there exists a unique element of $(\hat{L}_P\mathfrak{g})^\tau$ such that $\ll d_A, d_A \gg = 0$. Any two lifts differ by a multiple of K. Thus let d'_A be a second lift satisfying $\ll d'_A, d'_A \gg = 0$. But we now $d'_A = d_A + cK$. Then we get

$$\ll d'_A, d'_A \gg = \ll d_A + cK, d_A + cK \gg = 2c \ll K, d_A \gg = -2c$$

So we get $c = 0$ and $d_A = d'_A$.

(b) $\Rightarrow$ (c) The splitting is given by $\beta \mapsto \beta^\tau_A$ where β^τ_A is the unique lift such that $\ll \beta^\tau_A, d_A \gg = 0$. A similar argument as above shows that indeed β^τ_A is unique. The equivariance follows from the fact that $gd_{(\phi^{-1})^*A} = \phi_*(d_A)$ and the invariance of inner product allows us to move ϕ_* to β^τ_A.

(b) $\Leftarrow$ (c) if the inner product is nondegenerate. If $\ll -, - \gg$ is nondegenerate and $s : (LG)^\tau \to (\hat{L}G)^\tau$ is a splitting, then the property that $\ll K, d_A \gg = -1$ and $\ll s((L_P G)^\tau), d_A \gg = 0$ determines d_A uniquely.

□

Proposition 3.7. *Fix a connection A_0, then we have already noted the $L\mathfrak{g}$-torsor $\mathcal{A}$ can be identified with $L\mathfrak{g}$ through the map $\xi \mapsto A_0 + \xi d\theta$. The dependence of d_A on ξ is as follows: $d_{A_0+\xi d\theta} = d_{A_0} - \xi$.*

In this case, $\beta^\tau_{A_0+\xi d\theta}$ is given by $\beta^\tau_{A_0} - \ll \beta, \xi \gg K$.

Proof. $d_{A_0+\xi d\theta}$ is defined by the properties $\pi_*(d_{A_0+\xi d\theta}) = i$ where $\pi : P \times S^1 \to S^1$ is projection to the base and $(A_0 + \xi d\theta)(d_{A_0+\xi\theta}) = 0$, the pairing of vector fields with a 1-form.

But $d_{A_0} - \xi$ clearly satifies the first, since ξ is vertical. For the second property, note that $(A_0 + \xi d\theta)(d_{A_0} - \xi) = A(d_{A_0}) + \xi - \xi = 0$ where we've used that $d\theta$ annihilates ξ and that $d\theta(d_{A_0}) = 1$ and that $A_0(\xi) = \xi$, since ξ is vertical.

$\beta^\tau_{A_0+\xi d\theta}$ is characterised by $\ll \beta^\tau_{A_0+\xi d\theta}, d_{A_0} - \xi \gg = 0$. We claim that $\beta^\tau_{A_0+\xi d\theta}$ is given by $\beta^\tau_{A_0} - \ll \beta, \xi \gg K$. To see this, note that $\ll \beta^\tau_{A_0}, d_{A_0} \gg = 0$, $\ll - \ll \beta, \xi \gg K, d_A \gg = \ll \beta, \xi \gg$, $\ll \beta^\tau_{A_0}, -\xi \gg = - \ll \beta, \xi \gg$ and $\ll - \ll \beta, \xi \gg K, \xi \gg = 0$. □

3.3.2 Finite energy and positive energy representations

Sketch. For a connection A, $[d_A, -]$ behaves like an energy operator. The direct sum of its eigenspaces is the space of finite energy loops. A positive energy representation ρ has the property that d_A act as operators with discrete spectrum bounded below. For a positive energy representation, we can define a dense subspace of finite energy vectors on which finite energy loops act nicely. These will be the natural domain of the Dirac operator.

Details. Our analysis of loop groups will use the concept of energy of a loop, roughly the angular momentum of the wave packet the loop represents. The energy depends on the choice of a connection, because a connection determines which wave packets wrap around straight or what exactly the component of the angular momentum in the transversal direction to the rotation direction of the circle is.

Fix a connection A, then the space $(L\mathfrak{g}_\mathbb{C})_{fin}(A)$ of finite energy loops with respect to A will be constructed as a direct sum of eigenspaces of d_A. The details of this construction are of lesser importance to the ideas of the proof, but more important is the fact that the finite energy loops are a dense subspace of $(L\mathfrak{g}_\mathbb{C})_{fin}(A)$ in which energy element can be decomposed as a finite sum of eigenvectors of d_A. Secondly, the notion of finite energy lifts to $(LG)^\tau$.

Parallel transport with respect to A defines a map of the fiber above $1 \in S^1$, given by multiplication with an element of G_1. This is called the holonomy $hol : \mathcal{A} \to G_1$. In our case, hol assigns to $\beta d\theta$ the element $\exp(\int_{S^1} \beta d\theta) \in G$. Infinitesimally, we get a unitary map $\exp(-2\pi i S_A)$ of the complexification $\mathfrak{g}_\mathbb{C}$ of the fiber above $1 \in S^1$ of the adjoint bundle, where S_A is a self-adjoint linear operator with eigenvalues strictly between -1 and 1 (if these occur, we modify S_A to replace them with 0).

Proposition 3.8. *For $\xi \in \mathfrak{g}$, let $\xi_A \in L\mathfrak{g}$ be given by applying $\exp(i\theta S_A)$ to the parallel transport of ξ with respect to A. (Note that the parallel transport of ξ will not give a loop unless ξ is in the 0-eigenspace of S_A; ξ_A is always a loop.) Define $z^n \xi_A(\theta) := e^{in\theta}\xi_A(\theta)$, then $[d_A, z^n \xi_A] = iz^n(S_A + n)(\xi)_A$.*

Proof. The parallel transport of ξ, which we also denote ξ, is uniquely determined by $[d_A, \xi] = 0$. Now note that since d_A is a lift of $\frac{d}{d\theta}$, we hage $[d_A, z^n \xi_A] = inz^n \xi_A + z^n[d_A, \xi_A] = z^n(in + iS_a)\xi_A + [d_A, \xi]$. The last term is zero, hence we're done. □

Definition 3.9. $(L\mathfrak{g}_{\mathbb{C}})_{fin}(A)$ is the direct sum $\bigoplus_{n\in\mathbb{Z}} z^n \mathfrak{g}_{\mathbb{C}}$.

We now continue by defining the only notion of representation for which much is known [Pre86, chapter 9] and which is interesting [Pre86, remark (ii) after theorem 9.3.5].

Definition 3.10. Let $(LG)^\tau$ be an admissible graded central extension of LG and let $\rho : (LG)^\tau \to U(V)$ be a unitary representation on a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space such that $\mathbb{T}$ acts by scalars. (V, ρ) is called a positive energy representation of level τ if the following conditions holds:

(a) The representation ρ extends to a unitary representation $\rho : (\hat{L}G)^\tau \to U(V)$.

(b) For all $A \in \mathcal{A}$, the operator $\dot{\rho}(d_A)$ is a skew-adjoint operator iE_A such that E_A has discrete spectrum which is bounded below.

We denote the eigenspaces of E_A of energy e by $V_e(A)$. We now note some properties of the energy operator E_A for later use.

Proposition 3.11. *Suppose that ξ is an eigenvector of S_A of eigenvalue ϵ, then $[E_A, \dot{\rho}(z^n\xi_A)] = (n+\epsilon)\dot{\rho}(z^n\xi)$. This implies that $\dot{\rho}(z^n\xi_A)$ maps $V_e(A)$ to $V_{e+n+\epsilon}(A)$.*

Positive energy representations have great properties; they behave exactly like the representations of a compact Lie group. Most of these properties can be found in [Pre86, chapter 5, 9, 11].

Proposition 3.12. ***(a)*** *Positive energy representations are unitarizable and decomposable as $V = \bigoplus_e V(e)$ where each $V(e)$ is finite-dimensional. Here $V(e)$ are the eigenspaces of $\dot{\rho}(d_{A_0})$.*

(b) *If property (b) of definition 3.10 hold for one connection A, it holds for all elements of $\mathcal{A}$.*

(c) *Positive energy representations are completely reducible.*

(d) *Positive energy representations admit an (projective) intertwining action of $Diff^+(S^1)$.*

(e) *For each level τ there is a finite set of irreducible positive energy representations.*

(f) *The isomorphism classes of irreducible positive energy representations of level τ are parametrized by the set of antidominant weights.*

Fix a connection A, then we define $V_{fin}(A)$ as the direct sum of the eigenspaces $V_e(A)$ of $\dot{\rho}(d_A)$.

Proposition 3.13. *The decomposition $\bigoplus_e V_e(A)$ has the following properties:*

(a) *If ξ_A is a eigenvector of S_A with eigenvalue ϵ, then $\dot{\rho}((z^n\xi_A)^\tau_A)(V_e(A)) \subset V_{e+n+\epsilon}(A)$.*

(b) *If V is irreducible and Ω is a vector of lowest energy, then the vectors for $\dot{\rho}((z^{n_j}(\xi_j)_A)^\tau_A) \cdots \dot{\rho}((z^{n_1}(\xi_1)_A)^\tau_A)\Omega$ span a dense subspace of V.*

(c) *If V is finitely reducible, each $\bigoplus_{e<s} V_e(A)$ is finite dimensional for all $s \in \mathbb{R}$ and their dimension grows approximantely linearly with respect to s.*

(d) *If $A' = A + \beta d\theta$, then $E_{A'} = E_A + i\dot{\rho}(\beta^\tau_A) + \frac{\ll\beta,\beta\gg}{2}$.*

3.3.3 The spin representations and canonical central extension of LG

Sketch. The construction of an adjoint representation, using this to create a canonical Clifford central extension of LG and the construction of a spin representation proceed analogously to the finite dimensional case. The main difference is the inclusion of a polarization to keep track of positive and negative energy.

Details. The construction of the infinite dimensional analogous of Pin^c and S depend on the choice of a polarization of the Hilbert space. In our case the polarization will keep track of the positive and negative energy loops.

Definition 3.14. A complex structure on H is an orthogonal skew-adjoint operator $J : H \to H$ such that $J^2 = -1$. A polarizing structure is a Fredholm operator such that its restriction to the orthogonal complement ia complex structure and the extension $J : H \otimes \mathbb{C} \to H \otimes \mathbb{C}$ has non-zero eigenvalues $\{\pm i\}$ of infinite multiplicity.

A polarizing structure J_0 determines a polarization $\mathcal{J}$ consisting of compatible complex structures:

If $\dim \ker J_0$ **is even.** Then $\mathcal{J}$ consists of all complex structures J such that $J - J_0$ is Hilbert-Schmidt.

If $\dim \ker J_0$ **is odd.** Then $\mathcal{J}$ consists of all real skew-adjoint Fredholm operators which $\dim \ker J = 1$ and J a complex structure on $(\ker J)^\perp$ such that $J - J_0$ is Hilbert-Schmidt.

Lemma 3.15. *$\mathcal{J}$ is independent on the choice of J_0; any other $J \in \mathcal{J}$ will generate the same polarization.*

Proof. This follows trivially from the definition. □

Definition 3.16. The restricted orthogonal group $O_{\mathcal{J}}(H)$ is given by all operators $T \in O(H)$ such that $TJT^{-1} \in \mathcal{J}$ for all $J \in \mathcal{J}$.

Using the constructions described in [Pre86, chapter 12], we can define the group $Pin^c_{\mathcal{J}}(H)$, which acts irreducibly unitary of a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space $\mathcal{S}$ of spinors. Analogously to the finite dimensional case, this action comes from an Clifford multiplication action $H^* \to End(\mathcal{S})$. A summary of the properties of these constructions is the following theorem, which we state without proof:

Theorem 3.17. *Given a Hilbert space H and a polarization $\mathcal{J}$ there exists a group $Pin^c_{\mathcal{J}}(H)$ which fits in the graded central extension:*

$$1 \to \mathbb{T} \to Pin^c_{\mathcal{J}}(H) \to O_{\mathcal{J}}(H) \to 1$$

There is a unique irreducible unitary graded representation S of $Pin^c_{\mathcal{J}}$, whose action we denote by χ. $\mathcal{S}$ admits a Clifford multiplication $\gamma : H^* \to End(\mathcal{S})$ such that $\gamma(CoAd(g)(\mu)) = \chi(g)\gamma(\mu)\chi(g)^{-1}$ and $\gamma(\mu)$ is skew-hermitian for all $\mu \in H^*$ and $g \in Pin^c_{\mathcal{J}}$.

We turn again to loop groups. For each connection on $G \times S^1 \to S^1$, there is a canonical polarizing structure J_A on the L^2-completion H of $L\mathfrak{g}$. It is given on the dense subspace of finite energy loops by:

$$J_A = \begin{cases} 0 & \text{on } H_0(A) \\ \frac{d_A}{|e|} & \text{on } H_e(A) \end{cases}$$

To check that this is independent of the A, we prove the following lemma.

Lemma 3.18. *$J_A - J_{A'}$ is Hilbert-Schmidt.*

Proof. Let S_A denote the map

$$S_A = \begin{cases} 0 & \text{on } H_0(A) \\ \frac{1}{|e|} & \text{on } H_e(A) \end{cases}$$

which is Hilbert-Schmidt by the growth of the eigenvalues of d_A. Then $J_A = S_A d_A$ and we can write $J_A - J_{A'} = S_A d_A - S_{A'} d_A + S_{A'} d_A - S_{A'} d_{A'}$. The second term $S_{A'}(d_A - d_{A'})$ is Hilbert-Schmidt since the difference of d_A and $d_{A'}$ is bounded, being multiplication by an element of $L\mathfrak{g}$. Then first term is Hilbert-Schmidt, even though d_A is unbounded, since the difference of eigenvalues goes asymptotically as $\frac{1}{|e|^2}$. □

Thus we get a canonical polarization $\mathcal{J}_0$. For loop groups, there is a analogue of the adjoint representation of a Lie group on its Lie algebra:

Proposition 3.19. *There is a continuous homomorphism $LG \to O_{\mathcal{J}_0}(H)$.*

Proof. See [Pre86, section 6.3] for two different proofs. □

We can now proceed as before, taking the pullback of the central extension of $O_{\mathcal{J}_0}(H)$ to obtain a central extension of LG.

Definition 3.20. The graded central extension $(LG)^\sigma$ is obtained as $LG \times_{O_{\mathcal{J}_0}(H)} Pin^c_{\mathcal{J}_0}(H)$.

This graded central extension is admissible with the basic bilinear form $\int_{S^1}(\beta_1(s), \beta_2(s))ds$. This gives a splitting $L\mathfrak{g} \to (L\mathfrak{g})^\sigma$. Through $Pin^c_{\mathcal{J}_0}$, $(LG)^\sigma$ acts on $\mathcal{S}$.

Proposition 3.21. *$\mathcal{S}$ is a positive energy representation of LG. The minimal energy with respect to a fixed connection A can be made to be zero and then $\mathcal{S}_0(A)$ is an irreducible graded $Cl^c(\mathfrak{z}_A)$-module. Secondly Clifford multiplication by an element $\beta^* \in L\mathfrak{g}$ of energy e with respect to A lowers energy by e, i.e. $[E_A, \gamma(\beta^*)] = -e\gamma(\beta^*)$.*

The restriction $Z^\sigma_A \to Z_A$ is isomorphic to the central extension constructed in the finite dimensional case. If $A = A_0$, $Z_{A_0} = G$ and the induced splitting $\mathfrak{g} \to \mathfrak{g}^\sigma$ is the canonical splitting.

3.3.4 A family of Dirac operators on a loop group

Sketch. For each irreducible admissible positive energy representation, we can define a family of Dirac operators indexed by the connections $\mathcal{A}$. This family is analogous to the family in the finite dimensional case. However, it is general unbounded, which we control by dividing by $(1 - D_A^2)^{\frac{1}{2}}$. This gives us a class in twisted K-theory.

Details. We have gathered sufficient tools to define a family of Dirac operators on a loop group. There are some problems with defining an Clifford action of the 3-form Ω given $\Omega(\beta_1, \beta_2, \beta_3) = \ll [\beta_1, \beta_2], \beta_3 \gg$, but these are solved in [FHT07b, section 3.3], giving an operator $\gamma(\Omega)_A$ depending on a connection $A \in \mathcal{A}$. This operator preserves energy and is $(LG)^\sigma$-equivariant.

Let $(LG)^\tau$ be a central extension satisfying a nondegeneracy condition: $\ll -, - \gg$ is positive definite. Then let V be a finite reducible positive energy representation of $(LG)^{\tau-\sigma}$. Then $V \otimes S$ is a finite energy representation of $(LG)^\tau$. For each connection A, choose a Hilbert basis $\{e_p\}$ of H such that each basis vector has finite energy $E(e_p)$ with respect to energy E_A. By density of the finite energy vectors this is possible. Let e^p by the corresponding dual basis. Denote by $(R_p)_A$ the action of $e_p^{\tau-\sigma}$ lifted using the splitting $(L\mathfrak{g})^{\tau-\sigma} \to L\mathfrak{g}$ given by A.

Definition 3.22. We define a Dirac family $D(V)$ parametrized by $\mathcal{A}$ as linear operator on $(V \otimes S)_{fin}(A)$.

$$A \mapsto D_A := i\gamma^p(R_p)_A + \frac{i}{2}\gamma(\Omega)_A$$

This is well-defined because $(R_p)_A$ can be defined using the splitting of induced by A and because only a finite number of terms are nonzero on each finite energy vector (R_p raises energy, γ^p lowers it). The family of Dirac operators has formal properties similar to the famil yof Dirac operators in the compact Lie group case.

Proposition 3.23. *For each $A \in \mathcal{A}$, D_A is an odd skew-adjoint operator on $(V \otimes S)_{fin}(A)$. D_A preserves energy, i.e. $[E_A, D_A] = 0$ and the dependence of D_A on A is as follows: if $A = A_0 + \beta d\theta$, then $D_A = D_{A_0} + \gamma(\beta^*)$.*

Proof. That D_A is odd and skew-adjoint is trivial. To prove that it preserves energy, we note that we have already shown that $(R_p)_A$ raises energy by $E(e_p)$, i.e. $[E_A, (R_p)_A] = E(e_p)(R_p)_A$. The properties of the spin representation give us that Clifford multiplication γ^p decreases energy by $E(e_p)$, i.e. $[E_A, \gamma^p] = -E(e_p)\gamma^p$. Finally, the symmetry and $(LG)^\sigma$-invariance of the bilinear form $\int_{S^1}(\beta_1(s), \beta_2(s))ds$ for the extension $(LG)^\sigma$ tells us that $\gamma(\Omega)_A$ preserves enegy, i.e. $[E_A, \gamma(\Omega)_A] = 0$. This means that D_A preserves energy, i.e. $[E_A, D_A] = 0$.

For the dependence on the connection, note that the lift $e_p^{\tau-\sigma}$ changes to $e_p^{\tau-\sigma} - \ll \beta, \xi \gg K$. But K acts centrally as multiplication by i, so $i\gamma^p(R_p)_{A+\xi d\theta} = i\gamma^p(R_p)_A + \gamma^p \ll e_p, \xi \gg$ and the latter is equal to $\gamma(\beta^*)$. The term $\gamma(\Omega)_A$ doesn't change. □

However, D_A will in general be unbounded. To control this, note that D_A is odd skew-adjoint and D_A^2 is nonpositive.

Definition 3.24. We define a controlled Fredholm family $F(V) : \mathcal{A} \to B(V \otimes S)$.

$$A \mapsto F_A := D_A/(1 - D_A^2)^{\frac{1}{2}}$$

Proposition 3.25. *The operator F_A is bounded and the map $F(V)$ is $(LG)^\tau$-equivariant.*

Proof. This first can be easily seen from the boundedness of F_A on the dense subspace of finite energy vectors. The second is a direct consequence of $(LG)^\tau$-equivariance of D_A, which follows from $(LG)^\tau$-equivariance of $\gamma(\Omega)_A$, the compatibility $\chi(g)\gamma(\mu) = \gamma(CoAd(g)(\mu))\chi(g)$ and the fact that the splitting which determines $(R_p)_A$ is equivariant with respect to the action. □

Now comes the direct analogue of the large calculation we did for the Dirac operator in the finite dimensional case:

Proposition 3.26. *If V is an irreducible positive energy representation then F_A is an odd skew-adjoint Fredholm operator and $\ker F_A \subset (V \otimes S)_{e_{min}}(A)$.*

Proof. That F_A is bounded, odd and skew-adjoint is trivial. The last statement is completely analogous to the case of a compact group G. The steps are as follows:

- **(a)** Show $D_A^2 + 2E_A$ is multiplication by a real constant.
- **(b)** Show that D_A^2 is maximal on $(V \otimes S)_{e_{min}}(A) = V_{e_{min}} \otimes S_0(A)$.
- **(c)** Show that D_A^2 on $V_{e_{min}} \otimes S_0(A)$ reduces to the operator of a compact group G [FHT07b, proposition 4.12].

That it is Fredholm follows from the final statement, since by skew-adjointness the cokernel of F_A has the same dimension as the kernel of F_A. □

Hence we obtain a map $\Psi' : R^{\tau-\sigma}(L_P G) \to K^{\tau+\dim G}_{(LG)^\tau}(\mathcal{A})$. However, note that the twisting $\tau = (\mathcal{A} \to G_1, (LG)^\tau, \epsilon)$ where the local equivalence $\mathcal{A}//LG \to G_1//G$, where LG action on $\mathcal{A}$ by pullback of a connection under the induced diffeomorphism of $G \times S^1$ of an element of LG and G acts on G_1 by conjugation, is given by the holonomy $hol : \mathcal{A} \to G_1$. Hence, we can consider Ψ' instead as a map Ψ into $K_G^{\tau+\dim G}(G_1)$. The map $\Psi : R^{\tau-\sigma}(G) \to K_G^{\tau+\dim G}(G_1)$ assigns to an irreducible positive energy representation V the class $[F(V)]$. Then the following theorem holds:

Theorem 3.27. *Let G be a compact Lie group. Then there is an isomorphism of abelian groups:*

$$\boxed{\Psi : R^{\tau-\sigma}(G) \to K_G^{\tau+\dim G}(G_1)}$$

3.4 The twisted case and fractional loops

In the twisted case, we start with a non-trivial principal G-bundle P. We define the gauge groups or twisted loop groups as follows:

Definition 3.28. For G a compact Lie group and LG denote the space of G-equivariant diffeomorphisms of P convering the identity of S^1. Furthermore, let $\hat{L}G$ denote the group of G-equivariant diffeomorphisms of P convering a rigid rotation S^1.

Most of the stuff of the untwisted case works similarly, although the holonomy needs some modification. To a principal bundle we can assign a union of connected components of G, which is a G-space under conjugation. This union will be denoted by $G[P]$. It can be given in two ways:

- A principal G-bundle is classified up to isomorphism by a homotopy class in $[S^1, BG]$. These are exactly conjugacy classes in $\pi_1(BG)$, which can be identifed with $\pi_0(G)$.
- Let $\mathcal{A}_P$ be the affine space of all connections on P. Fix a fiber and a $p \in P$ in this fiber. Then parallel transport over the circle gives us the holonomy map $hol : \mathcal{A}_P \to G$, which assigns to each connection its holonomy with respect to p. Its image is $G[P]$.

A consequence of the first remark is that all principal G-bundles over the circle are trivial if G is connected and intuitively all principal G-bundles over the circle come from permuting components.

The rest of the theory was set up in such a way that the theory remains almost the same. This would have been different if we singled out the standard connection A_0 to identify the $L\mathfrak{g}$-torsor $\mathcal{A}$ with $L\mathfrak{g}$.

A second generalization is to replace the $\mathbb{T}_{rot}$ appearing in the definition of the extended loop group $\hat{L}G$ with a finite cover $\tilde{\mathbb{T}}_{rot}$ of $\mathbb{T}_{rot}$ of finite degree $n \in \mathbb{Z}$: the new definition of $\hat{L}_n G$ is that it is the pullback $\hat{L}G_{\mathbb{T}_{rot}}\tilde{\mathbb{T}}_{rot}$. The results require no modifications, but this generalization is useful in the case of tori, because a double cover occurs in the proof.

A Prerequisites

A.1 Clifford algebras and spinor representations

A.1.1 Clifford algebras

Let $\mathbb{F}$ denote $\mathbb{R}$ or $\mathbb{C}$ and let V be a $\mathbb{F}$-vector space . We denote by $T(V)$, $\Lambda(V)$ and $S(V)$ are the tensor algebra, exterior algebra and symmetric algebra on a $\mathbb{F}$-vector space V respectively. Usually $\Lambda(V)$ and $S(V)$ are constructed as quotients of $T(V)$. Alternatively, these algebras can be defined using universal properties. For example, if $U : \mathbb{F}\mathsf{Alg} \to \mathbb{F}\mathsf{Vect}$ is the forgetful functor, then $T(V)$ is the universal $\mathbb{F}$-algebra with embedding $i_V : V \to U(T(V))$ such that every map of $\mathbb{F}$-vector spaces $V \to U(A)$ where A is a $\mathbb{F}$-algebra factors as $U(f) \circ i$ for a map $f : T(V) \to A$ of $\mathbb{F}$-algebras:

$$\begin{array}{ccccc} V & \xrightarrow{i} & U(T(V)) & \qquad & T(V) \\ & \searrow & \downarrow{\scriptstyle U(f)} & & \downarrow{\scriptstyle f} \\ & & U(A) & & A \end{array}$$

$\Lambda(V)$ and $S(V)$ satisfy similar universal properties for antisymmetric and symmetric $\mathbb{F}$-algebras respectively.

Alternatively, one can define these algebras in terms of a basis $e_1, \ldots, e_n$ of V. For example, $\Lambda(V)$ is generated by $e_1, \ldots, e_n$ subject to the relation $e_i e_j + e_j e_i = 0$.

Definition A.1 (Clifford algebra)**.** Let V be a finite dimensional vector space with bilinear form $(-,-)$. Then $Cl(V)$ is the $\mathbb{R}$-algebra $T(V)/I$ where I is the ideal generated by $\{v \otimes w - (v,w)1 | v, w \in V\}$.

From now on, we'll assume $(-,-)$ is non-degenerate. There are several alternative constructions, all of which give the same $\mathbb{F}$-algebra. Furthermore, $Cl(V)$ can be described as the solution to an universal problem. In terms of a basis $e_1, \ldots, e_n$, it is generated by $e_1, \ldots, e_n$ subject to the relation $e_i e_j + e_j e_i = 2(e_i, e_j)$. Clifford algebras have several important features:

Relation with $\Lambda(V)$. Note that $\Lambda(V)$ is $Cl(V)$ for V with zero bilinear form. It is then easy to see that $\Lambda(V)$ and $Cl(V)$ are isomorphic as $\mathbb{F}$-vector spaces and we can conclude $\dim Cl(V) = 2^{\dim V}$.

$\mathbb{Z}/2\mathbb{Z}$-grading. The ideal generated by $\{v \otimes w - (v,w)1 | v, w \in V\}$ is concentrated in even degrees. Therefore, the $\mathbb{Z}_{\geq 0}$-grading induces a $\mathbb{Z}/2\mathbb{Z}$-grading on $Cl(V)$.

(Anti-)automorphisms. The Clifford algebra has several (anti)automorphisms, which are induced from (anti)automorphisms of $T(V)$. The transpose $(-)^t$ is induced by the map of $T(V)$ which reverses ordering, i.e. maps an element of the form $v_1 \otimes v_2 \otimes \ldots \otimes v_k$ to $v_k \otimes \ldots \otimes v_2 \otimes v_1$.

The automorphism of $T(V)$ given by $v \mapsto -v$ on generators, induces an automorphism α of $Cl(V)$. This is called the grading involution. The composition of the transpose and the grading involution $x \mapsto \bar{x} := \alpha(x^t)$ is again an antiautomorphism of $Cl(V)$.

Functoriality The construction of Clifford algebras is functorial in V. Any linear map $f : V \to W$ preserving the inner product induces a map $f^* : Cl(V) \to Cl(W)$.

There are several varieties of Clifford algebra associated to a real vector space. There are two $\mathbb{R}$-algebras: Applying the Clifford algebra construction to V with bilinear form $(-,-)$ gives $Cl^+(V)$, applying the Clifford algebra construction to V with bilinear form $-(-,-)$ gives $Cl^-(V)$. However, the important case for us will the complexication of a Clifford precise. To be precise, we will define $Cl^c(V)$ as $Cl^-(V) \otimes \mathbb{C}$. It is isomorphic to $Cl(V \otimes \mathbb{C})$ if we use the bilinear extension of $(-,-)$, not the hermitian one.

A.1.2 *Pin*^{c}

We can make the earlier linear isomorphism between $\Lambda(V)$ and $Cl(V^*) \cong Cl(V)$ (where V^* has the dual inner product) more explicit. There is a filtration of $Cl(V^*)$ induced by multiplication with generators. The associated graded algebra of $Cl(V^*)$ is then $\Lambda(V)$.

Let $\{e_a\}$ be a basis of V and $\{e^a\}$ be the dual basis. Let $g_{ab} = (e_a, e_b)$ denote the metric tensor and $g^{ab} = (e^a, e^b)$ the dual metric tensor. Using the filtration $\Lambda^2(V)$ can be identified with the linear span of $\frac{1}{4}(e^a e^b - e^b e^a)$ for $1 \leq i < j \leq n$. To be precise, we use the somewhat unusual identification $e_a \wedge e_b \mapsto \frac{1}{4}(e^a e^b - e^b e^a)$. In fact $\Lambda^2(V)$ is a Lie algebra and the reason for the factor $\frac{1}{4}$ is that it is necessary to get a basis of a Lie algebra which is closed under the Lie bracket.

Proposition A.2. $\Lambda^2(V) \subset Cl(V^*)$ *is a Lie algebra under the graded commutator.*

Proof. It suffices to calculate the commutator of two basis elements of the form $\frac{1}{2}e^a e^b$:

$$[\frac{1}{4}(e^a e^b - e^b e^a), \frac{1}{4}(e^c e^d - e^d e^c)] =$$

$$g^{ac}\frac{1}{4}(e^b e^d - e^d e^b) + g^{bc}\frac{1}{4}(e^a e^d - e^d e^a) + g^{ad}\frac{1}{4}(e^b e^c - e^c e^b) + g^{bd}\frac{1}{4}(e^a e^c - e^c e^a)$$

This also shows that the image of the bracket is contained in $\Lambda^2(V)$. Antisymmetry and the Jacobi identity are clear from the definition of the bracket as coming from the commutator. □

In terms of a basis of V the Lie algebra $\mathfrak{so}(V)$ of skew-symmetric matrices is generated by matrices E_{ab} for $1 \leq a < b \leq \dim V$, given by $g_{cb}(e^a \otimes e^c - e^c \otimes e^a)$. If the basis is orthonormal, then this matrix has components which are zero except for the (a,b)'th entry, which is 1, and the

(b,a)'th entry, which is -1, and we get the usual generators of $\mathfrak{so}(V)$. These satisfy the commutation relations

$$[E_{ab}, E_{cd}] = g^{ac}E_{bd} + g^{bc}E_{ad} + g^{ad}E_{bc} + g^{bd}E_{ac}$$

There is a isomorphism of vector spaces between $\mathfrak{so}(V)$ and $\Lambda^2(V)$, mapping E_{ab} to $e_a \wedge e_b$, which gets identified with $\frac{1}{4}(e^a e^b - e^b e^a) \in \Lambda^2(V) \subset Cl(V^*)$. Our earlier calculations show this is an isomorphism of Lie algebras.

If we want to work coordinate-invariantly, like we should, then the isomorphism of Lie algebras is implemented by the composition of the following two maps:

$$A \ni \mathfrak{so}(V) \mapsto (A-,-) \in \Lambda^2(V)$$

$$\alpha \ni \Lambda^2(V) \mapsto \mu(i \otimes i)(\alpha) \in Cl(V^*)$$

where in the last line μ is Clifford multiplication and i in the inclusion $V^* \hookrightarrow Cl(V^*)$ and we consider $\Lambda^2(V) \hookrightarrow V^* \otimes V^*$ with balanced representative, i.e. $a \wedge b \mapsto \frac{1}{2}(a \wedge b - b \wedge a)$.

Then we define the Lie groups $Spin^{\pm}(V)$ as the groups given by exponentiating $\Lambda^2(V)$ in $Cl^{\pm}(V^*)$. These form a double cover of $SO(V)$, which can extended to a double cover of $O(V)$, giving the Lie groups $Pin^{\pm}(V)$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & Spin^{\pm}(V) & \longrightarrow & SO(V) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & Pin^{\pm}(V) & \longrightarrow & O(V) & \longrightarrow & 1 \end{array}$$

Similarly, we obtain $Pin^c(V) \subset Cl^c(V^*)$. This is a central extension of $Pin^-(V)$ by $\mathbb{T}$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & Pin^{-}(V) & \longrightarrow & O(V) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{T} & \longrightarrow & Pin^{c}(V) & \longrightarrow & O(V) & \longrightarrow & 1 \end{array}$$

Alternatively definitions of the Pin groups abound. Two well-known alternatives are the following: one can take Pin to be the invertible elements in the appropiate Clifford algebra such that $xv\alpha(x)^{-1} \in V$ for all $v \in V$ and $x^t x = 1$. Using the Pfaffian of a skew-symmetric map, one can define the spin group as a element A of $O(V \otimes \mathbb{C})$ together with a square root of a polynomial defined on a chart of the space of complex structures.

A.1.3 The spinor representation

We want to a construct an irreducible $\mathbb{Z}/2\mathbb{Z}$-graded $Cl^c(V)$-module for later use. To do this we first note that there is a $Cl^c(V)$-module structure on $\Lambda(V) \otimes \mathbb{C}$. There are two main constructions:

As a subrepresentation of $\Lambda(V) \otimes \mathbb{C}$ The idea is that the action of $Cl^c(V)$ on V extends to an action $\Lambda(V)$, which is reducible. By choosing any complex structure on V, we can construct a representation of $Cl^c(V)$ on $\Lambda(W)$, where W the $+i$ eigenspace of J in $V \otimes \mathbb{C}$. Generators of $Cl^c(V)$ coming from W will act as creation operators, generators coming from $\bar{W}$ will act as annihilation operators. This representation is irreducible. For more information, look at [Woi08, section 2.3] or [Pre86].

As holomorphic sections of a line bundle We can define a holomorphic line bundles Pf over $\mathcal{J}(V)$, the space of complex structures on V, which can be identified with a submanifold over $Gr(V)$ by representing a complex structure by its isotropic i-eigenspace. We can define the spinor representation as the holomorphic section $\Gamma(Pf^*)$ of the dual bundle to Pf.

If $\dim V$ is odd, there is a single irreducible $\mathbb{Z}/2\mathbb{Z}$-graded $Cl^c(V)$ representation. It has a commuting action of $Cl^c(1)$, which we consider part of the structure. If $\dim V$ is even, there are two irreducible $\mathbb{Z}/2\mathbb{Z}$-graded $Cl^c(V)$ representations. These are distinguished by the action of the volume form. We fix one of them.

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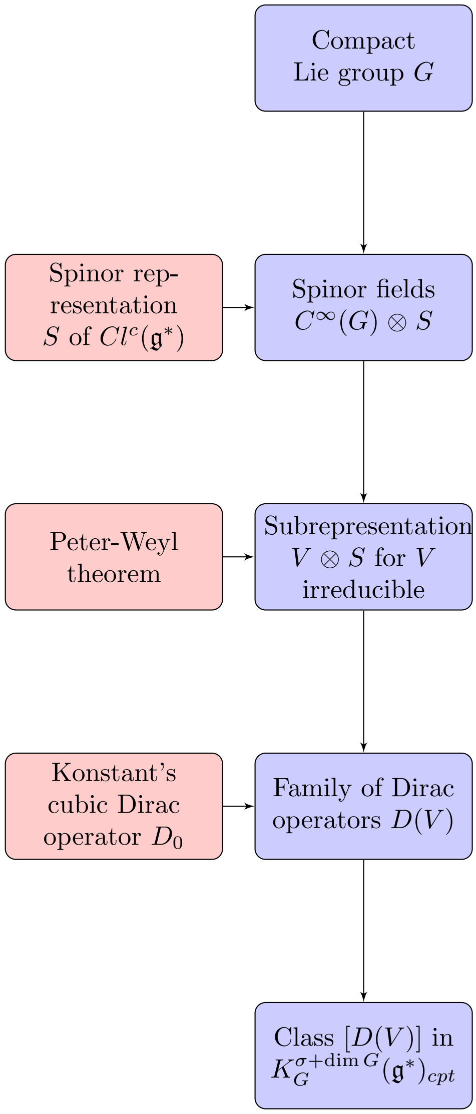

Figure 1: An figure of the construction of K-classes for irreducible representations of a compact Lie group.

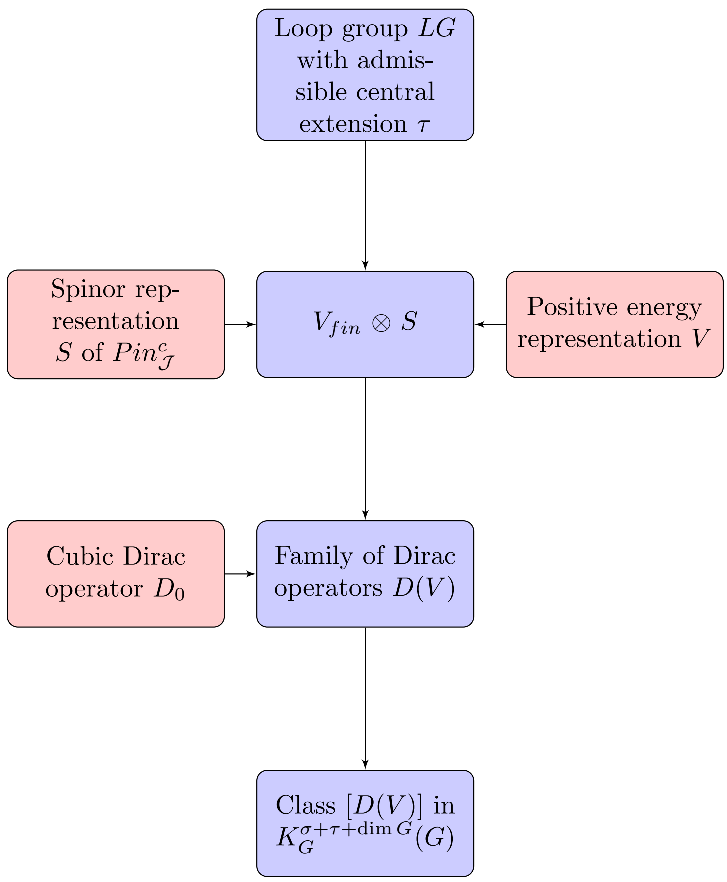

Figure 2: An figure of the construction of K-classes for irreducible positive energy representations at leavel τ of a loop group with admissible central extesnsion τ.