1. Character Formulas, Dario Beraldo, UC Berkeley

Weyl character formula for compact Lie groups. Let G be a compact Lie group and fix a maximal torus $T \subset G$, with Lie algebra $\mathfrak{t}$. Inside $i\mathfrak{t}^*$ we have some lattices:

$$\text{roots of } \mathfrak{g} \subset Hom(T, S^1) \subset \text{ weights of } \mathfrak{g}$$

Here we consider $Hom(T, S^1) \subset i\mathfrak{t}^*$ as follows: there is an isomorphism ($L := \ker \exp_{|\mathfrak{t}}$)

$$exp : \mathfrak{t}/L \xrightarrow{\cong} T.$$

Defining

$$\Lambda := \left(\frac{L}{2\pi i}\right)^* = \{\phi \in i\mathfrak{t}^* \,|\, \phi(L) \subset 2\pi i\mathbb{Z}\},$$

we see that Λ corresponds to $Hom(T, S^1)$ via exponentiation. The key point is that, for $\phi \in \Lambda$, the following function on T is well-defined: $e^\phi : \exp(X) \mapsto e^{\phi(X)}$.

It is well-known that irreducible reps of G are in bijection with Λ_+ (the intersection of Λ with the positive Weyl chamber), where $\lambda \in \Lambda_+$ maps to the irrep (π_λ, V_λ) with highest weight λ. The Weyl character formula for $X \in \mathfrak{t}$ computes that trace of $\pi_\lambda(e^X)$:

$$\chi_\lambda(exp(X)) := tr(\pi_\lambda(\exp X)) = \frac{\sum_w \epsilon(w) e^{(w\cdot(\lambda+\rho))(X)}}{e^{\rho(X)} \prod_{\alpha>0}(1 - e^{-\alpha(X)})},$$

where $\rho = \frac{1}{2}\sum_{\alpha>0} \alpha = \sum_{i=1}^{\ell} w_i$ is the Weyl vector. The denominator of this formula is independent of λ and is called the Weyl denominator.

Remark: in general $w \cdot (\lambda + \rho)$ and ρ don't belong to Λ, so that $e^w \cdot (\lambda + \rho)$ and e^ρ are not well defined, but their ratio is (as $w(\rho) - \rho$ is always in the root lattice).

Kirillov character formula for compact Lie groups. Now we will explain Kirillov's orbit method philosophy. Let G any Lie group (not necessarily compact). We know that G acts on its Lie algebra $\mathfrak{g}$ by the adjoint action; dualizing we get an action of G on $\mathfrak{g}^*$, the coadjoint action, denoted by K.

$$g \cdot \phi = K(g)\phi = \phi \circ Ad(g^{-1}). \quad (\phi \in \mathfrak{g}^*, g \in G).$$

It's also worthwhile to write down the action K_* of $\mathfrak{g}$ on its dual:

$$K_*(X)\phi = \phi \circ (-ad(X)) \quad (\phi \in \mathfrak{g}^*, X \in \mathfrak{g}).$$

Now, $\mathfrak{g}^*$ is partitioned into coadjoint orbits and Kirillov's philosophy says: *the set of coadjoint orbits in $\mathfrak{g}^*$ is in bijection with the set of unitary irreducible representations of G.*

The way of attaching to Ω (a coadjoint orbit) a unitary irrep π_Ω may go under the name of quantization. Indeed, the key property of coadjoint orbits is the following:

Proposition 1.1. *Each coadjoint orbit is a homogeneous symplectic manifold.*

Then the orbit Ω, being symplectic, may be regarded as the classical phase space of a physical system with symmetry G. Quantizing, one gets a quantum phase space, namely a unitary representation π_Ω of G on an Hilbert space. The fact that Ω is homogeneous should correspond to the irreducibility of π_Ω.

Unfortunately, this doesn't work for all Lie groups, for example the above is false for a compact Lie group, as we will see. However this does work if G is 1-connected with nilpotent Lie algebra.

Using the symplectic form, denoted σ, we can define a measure on Ω, namely the volume form $\sigma^d/d!$, and Kirillov character formula says that

$$\chi_{\pi_\Omega} = \hat{\mu_\Omega}.$$

Again, the character is the trace of the representation evaluated on the exponential of a Lie algebra element.

Before looking at an example, let me explain Proposition 1.1. Picking a point $\phi \in \Omega$, we get the following isomorphism

$$G/Stab_\phi \to \Omega$$

$$g \mapsto K(g)\phi.$$

Since $Stab_\phi = \{g \,|\, K(g)\phi = \phi\}$ is a closed group of G, each orbit is G-homogeneous.

Now we wish to define a form σ (that will turn out to be symplectic). By homogeneity, we just need to define a skew-symmetric bilinear form on $T_\phi\Omega$, invariant under $Stab_\phi$. Clearly,

$$T_\phi\Omega = \mathfrak{g}/s_\phi$$

where

$$s_\phi = Lie(Stab_\phi) = \{X \in \mathfrak{g} | K_*(X)\phi = 0\}.$$

Then

$$\sigma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$$

$$(X, Y) \mapsto \phi([X, Y]).$$

By the very definition of K_*, one sees that $\ker \sigma = s_\phi$, so that σ descends to a non-degenerate form on $T_\phi\Omega$. Furthermore, one can easily show that σ is closed using the Jacobi identity. This completes the proof of the proposition.

For an illustration of the orbit philosophy, let's consider the Heisenberg group of 3×3 upper triangular matices with 1's on the diagonal:

$$H := \{g_{abc} := \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} | a, b, c \in \mathbb{R}\}.$$

It is simply connected and nilpotent.

First, let me describe the irreducible representations of H: they are either infinite-dimensional, denoted π_λ with $\lambda \in \mathbb{R}^*$, or one-dimensional, $\pi_{\mu,\nu}$ with $(\mu, \nu) \in \mathbb{R}^2$.

Here are the definitions:

$$\pi_\lambda(g_{abc}) : L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}, dx)$$

with

$$(\pi_\lambda(g_{abc})f)(x) = e^{2\pi\lambda i(bx+c)} f(x + a)$$

(this rep is obviously unitary and it is irreducible by Wiener's theorem).

$$\pi_{\mu,\nu}(g_{abc}) = e^{2\pi i(a\mu+b\nu)}.$$

Accepting this, let's compute the coadjoint orbits. Using the non-degenerate pairing

$$\langle A, B\rangle = tr(AB),$$

we identify $\mathfrak{g}^* = Mat(3 \times 3)/\mathfrak{g}^\perp$, which is $Mat(3 \times 3)$ modulo the entries on and above the main diagonal. Then cadjoint orbits are equivalent to adjoint ones, and we compute

$$K(g_{abc}) \begin{pmatrix} * & * & * \\ x & * & * \\ z & y & * \end{pmatrix} = g_{abc} \begin{pmatrix} * & * & * \\ x & * & * \\ z & y & * \end{pmatrix} g_{abc}^{-1} = \begin{pmatrix} * & * & * \\ x + bz & * & * \\ z & y - az & * \end{pmatrix}$$

Thus we see that the coordinate z is fixed on the orbit; let $z = \lambda$. If $\lambda \neq 0$, by changing a, b we notice that the orbit is the entire plane $\{z = \lambda\}$ in $\mathfrak{g}^* \cong \mathbb{R}^3$. On the contrary, if $\lambda = 0$, orbits are points $\{(\mu, \nu, 0)\}$.

This suggests the correspondence of $\Omega := \Omega_\lambda = \{z = \lambda\}$ and π_λ; let's verify the character formula in this case.

We have a measure on these orbits, given by the symplectic volume (I omit the short calculation),

$$\mu_\Omega := \sigma = \frac{dx \wedge dy}{\lambda}$$

and need to verify

$$\chi_{\pi_\Omega} = \hat{\mu_\Omega},$$

in distributional sense (since we are dealing with an infinite-dimensional rep), namely $(\chi_{\pi_\Omega}, \phi) = (\hat{\mu}_\Omega, \phi) = (\hat{\phi}, \mu_\Omega)$.

Given a test function $\phi \in S(G)$ (G is the Heisenberg group) we define the operator (it's π_λ weighted by ϕ)

$$\pi_\lambda(\phi) = (\pi_\lambda(g), \phi) = \int_G \pi_\lambda(g_{abc})\phi(a, b, c)dadbdc.$$

Now,

$$
\begin{aligned}
(\pi_\lambda(\phi)(f))(x) &= \int_{\mathbb{R}^3} \phi(a,b,c)e^{2\pi i\lambda(bx+c)}f(x+a)dadbdc \\
&= \int_{\mathbb{R}^3} \phi(y-x,b,c)e^{2\pi i\lambda(bx+c)}f(y)dydbdc \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{\phi}(y-x,\lambda x,c)e^{2\pi i\lambda c}f(y)dcdy \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \hat{\phi}(y-x,\lambda x,c)e^{2\pi i\lambda c}dc\right) f(y)dy \\
&= \int_{\mathbb{R}} K(x,y)f(y)dy,
\end{aligned}
$$

where $K(x,y) = \left(\int_{\mathbb{R}} \hat{\phi}(y-x,\lambda x,c)e^{2\pi i\lambda c}dc\right)$.

Since $\hat{\phi}$ is Schwartz, this operator is trace class and the trace is obtained integrating the kernel K along the diagonal:

$$
tr\pi_\lambda(\phi) = \int_{\mathbb{R}} K(x,x)dx = \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{\phi}(0,\lambda x,c)e^{2\pi i\lambda c}dcdx =
$$

$$
\frac{1}{\lambda}\int_{\mathbb{R}}\int_{\mathbb{R}} \hat{\phi}(0,u,c)e^{2\pi i\lambda c}dcdu = \int_{\mathbb{R}} \phi(0,0,c)\frac{e^{2\pi i\lambda c}}{\lambda}dc,
$$

Thus

$$
\chi_{\pi_\Omega} = \frac{e^{2\pi i\lambda c}}{\lambda}\delta(a)\delta(b).
$$

On the other hand,

$$
\begin{aligned}
(\hat{\mu_\Omega},\phi) = (\mu_\Omega,\hat{\phi}) &= \int_{\Omega_\lambda}\left(\int_{\mathbb{R}^3} \phi(a,b,c)e^{2\pi i(ax+by+cz)}dadbdc\right)\frac{dx\wedge dy}{\lambda} \\
&= \int_{\mathbb{R}^3}\left(\int_{\Omega_\lambda} \phi(a,b,c)e^{2\pi i(ax+by+c\lambda)}\frac{dxdy}{\lambda}\right)dadbdc \\
&= \int \phi(a,b,c)\left(\int e^{2\pi iax}dx\right)\left(\int e^{2\pi iby}dy\right)\frac{e^{2\pi icz}}{\lambda}dadbdc \\
&= \int \phi(a,b,c)\delta(a)\delta(b)\frac{e^{2\pi icz}}{\lambda}dadbdc,
\end{aligned}
$$

in accordance with Kirillov's formula.

For compact groups, Kirillov's character formula needs to be modified in three ways. First, there are too many coadjoint orbits and too few irreducible representations of G. Namely, using the Killing form on $\mathfrak{g}$, we know that coadjoint orbits are identified with coadjoint orbits. Now, adjoint orbits are flag varieties, G/Γ, for Γ conjugated to a subgroup containing T, and it is well-known that each orbit intersects the positive Weyl chamber in exactly one point. So, coadjoint orbits are parametrized by the positive Weyl chamber, while unirreps are parametrized by the discrete set Λ_+.

Another problem is the more complicated (compared to nilpotent groups) nature of the map $exp : \mathfrak{g} \to G$, that forces to introduce a distorsion factor

$$p(X) = \frac{\sinh(ad\ X/2)}{ad\ X/2}$$

in the character formula. A third problem is a ρ-shift.

As an example, let's look at $SU(2)$. Coadjoint orbits are flag varieties, i.e. spheres, which are centered at $0 \in \mathfrak{su}(2)$. So let Ω_r be the sphere of radius r, and, after passing to spherical cordinates, $\sigma = r \sin\phi d\theta \wedge d\phi$ (this is a cumbersome computation). Let $Z = diag(iz, -iz) \in \mathfrak{su}_2$. Then

$$\int_{\Omega_r} e^{2\pi i <Z,F>} d\sigma(F) = \frac{\sin rz}{z}$$

Dividing by

$$p(Z) = \frac{\sinh(ad\ Z/2)}{ad\ Z/2} = \frac{\sin z}{z}$$

we get

$$\frac{1}{p(Z)} \int e^{2\pi i <Z,F>} d\sigma(F) = \frac{1}{p(Z)} \frac{\sin rz}{z} = \frac{\sin rz}{\sin z}.$$

Now notice that, if $r = n + 1$, the RHS is exactly the character of the representation of $SU(2)$ corresponing to the weight n. We see here the ρ-shift, as $\rho \cong 1$ in this case.

To summarize, the Kirillov character formula (for G compact):

Theorem 1.2 (Kirillov)**.** *For $\lambda \in \Lambda_+$ and $X \in \mathfrak{t}$, we have*

$$\chi_\lambda(exp(X)) = \frac{1}{p(X)} \int_{\Omega_{\lambda+\rho}} e^{2\pi i <X,F>} d\mu(F)$$

Loop groups. Both formulas admit generalizations to the case of loop groups. We have a central extension of $L\mathfrak{g}$

$$\tilde{L\mathfrak{g}} \cong L\mathfrak{g} \oplus \mathbb{R}C$$

and we also define

$$\hat{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{R}C \oplus \mathbb{R}D$$

with

$$[D, X \otimes z^n] = nX \otimes z^n, \quad [D, C] = 0.$$

We have an invariant bilinear form defined by

$$< D, C >= 1, \quad < X \otimes z^m, Y \otimes z^n >=< X, Y > \delta_{m,-n}, \ \text{the other products are zero.}$$

In principle we want to calculate the action of $\hat{LG}$ acting on $\hat{L\mathfrak{g}}^*$, but we'll just do LG acting on $\tilde{L\mathfrak{g}}^*$. Suitably interpreted, this action has an intrinsic geometric meaning. We have the sequence

$$0 \to \mathbb{R}C \to \tilde{L\mathfrak{g}} \to L\mathfrak{g} \to 0$$

and its dual

$$0 \to (L\mathfrak{g})^* \to \tilde{L\mathfrak{g}}^* \to \mathbb{R}\delta \to 0$$

where δ is an element determined by $\delta : C \mapsto 1$. Then

$$(\tilde{L\mathfrak{g}})^*_\ell := \{\phi : \tilde{L\mathfrak{g}} \to \mathbb{R} | \phi(C) = \ell\} = l\delta + (L\mathfrak{g})^*.$$

Moreover,

$$(L\mathfrak{g})^* = \Omega^1(S^1; \mathfrak{g}^*)$$

via the pairing $(X \otimes f, Y \otimes \omega) =< X, Y > \int f\omega$. Using the Killing form, we finally obtain

$$(L\mathfrak{g})^* = \Omega^1(S^1; \mathfrak{g}^*) = \Omega^1(S^1; \mathfrak{g}),$$

so that $(\tilde{L\mathfrak{g}})^*_\ell = l\delta + \Omega^1(S^1; \mathfrak{g})$, which is isomorphic to the space of connections on the trivial principle G bundle over S^1.

Let's now describe the coadjoint action on $(\tilde{L\mathfrak{g}})^*_\ell$, interpreted as a space of connections.

Let $x, y \in L\mathfrak{g}$, $\omega \in \Omega^1(\mathfrak{g})$, $\alpha \in \mathbb{R}$:

$$\begin{aligned} < K_*(x)(\ell\delta + \omega), y + \alpha C > &= - < \ell\delta + \omega, [x, y + \alpha C] > \\ &= - < \ell\delta + \omega, [x, y] > \\ &= -\ell < dx, y > - < \omega, [x, y] > \\ &= -\ell < dx, y > + < [x, \omega], y > \\ &=< -\ell dx + [x, \omega], y + \alpha C > \end{aligned}$$

so we find that the coadjoint action corresponds to the infinitesimal gauge action

$$x \cdot (\ell\delta + \omega) = -\ell dx + [x, \omega] = [x, \ell d + \omega]$$

So LG acts on $(L\mathfrak{g})^*_\ell$ by $\gamma \cdot (d + \omega) = \gamma(d + \omega)\gamma^{-1}$, and we see

$$(L\mathfrak{g})^*_\ell / LG \cong \mathcal{A}/LG = G//G$$

1.1. **Kirillov character formula for $\hat{LG}$.** I. Frenkel has extended Kirillov character formula to loop groups. Let λ be a dominant weight of LG. Then

$$\chi_\lambda(exp(bD + Y)) = \frac{1}{p(bD + Y)} e^{b/a(H,H)} \int_{\Omega_{\lambda+\hat{\rho}}} e^{2\pi i <bD+Y,F>} d\mu^{a/b}(F).$$

Here$\hat{\rho}$ is the Kac-Weyl vector to be defined below, p is a function depending only on G and $\mu^{a/b}$ is a particular measure on the orbit $\Omega_{\lambda+\hat{\rho}}$ (that I won't describe). We also need to know what H and a are.

This formula is defined only when $Im(b) < 0$; indeed only in this case the exponential inside the integral is rapidly descreasing, making the integral converge.

Recall that the roots of $\hat{LG}$ are

$$\{R_\mathfrak{g} + \mathbb{Z}C\} \bigsqcup (\mathbb{Z} - \{0\})C,$$

where $R_\mathfrak{g}$ are the roots of G.

The simple roots are $\hat{\alpha}_1 := \alpha_1, \ldots, \hat{\alpha}_\ell := \alpha_\ell$ and $\hat{\alpha}_0 := C - \theta$. In other words, the simple roots for $\hat{L}G$ are the simple roots of G, plus $\hat{\alpha}_0$. Here θ

is the top root of G. The fundamental weights, denoted $\{\hat{\omega}_0, \ldots, \hat{\omega}_\ell\}$, must satisfy the following defining relations:

$$\frac{2(\hat{\omega}_i, \hat{\alpha}_j)}{(\hat{\alpha}_j, \hat{\alpha}_j)} = \delta_{ij}.$$

The generalized Weyl vector is defined to be

$$\hat{\rho} = \sum_{i=0}^{\ell} \hat{\omega}_i$$

and it can be computed explicitely: $\hat{\rho} = \frac{D}{2} + \rho$ (here ρ is the Weyl vector of G). Now, we can uniquely write

$$\lambda + \hat{\rho} = aD + H$$

for some $a > 0$ and $H \in \mathfrak{t} \otimes \mathbb{C}$.