1. K-Theory of Topological Stacks, Ryan Grady, Notre Dame

Throughout, G is sufficiently "nice:" simple, maybe π_1 is free, or perhaps it's even simply connected. Anyway, there are some assumptions lurking. The reference for the following is [?] and [?]. As motivation, consider the following black-box theorem:

Theorem 1.1.

$$^{k}R(LG) \cong {}^{k+n}K_G^{dim\ G}(G) \cong {}^{k+n}K(G//G) \cong {}^{k+n}K(A/LG)$$

where A is some space of connections (with values in $\mathfrak{g}$) on the trivial principle bundle over S^1, and LG acts by gauge transformations.

Now we would like to consider S^1 acting by loop rotations (i.e. lifting the action on the base to the bundle) and denote the resulting groupoid by

$$A/S^1 \rtimes LG.$$

We then have an analogous theorem.

Theorem 1.2.

$$^{\tau-\sigma}R(S^1 \rtimes LG) \cong {}^{\tau}K^{dim\ G}(A/S^1 \rtimes LG)$$

The groupoid $A/S^1 \rtimes LG$ is an example of a local (but not global) quotient groupoid and hence we need to develop the K-theory of such objects.

1.1. Topological Groupoids.

Definition 1.3. A topological groupoid is a pair of spaces (X_0, X_1) with source and target morphisms, $s, t : X_1 \to X_0$, and identity section $X_0 \to X_1$, an inverse $inv : X_1 \to X_1$ and a composition, $c : X_1 \times_{X_0} X_1 \to X_1$.

Examples 1.4.

- If X is a G-space, $X \times G \rightrightarrows X$ is a topological groupoid, denoted $X//G$.
- If X is a space and $\mathcal{U} = \{U_i\}$ is a cover, then define a topolgical groupoid $N_{\mathcal{U}}$ whose objects are pairs (U_i, x_i), $x_i \in U_i$ and a morphism from (U_i, x_i) to (U_j, x_j) is an ω in $U_i \times_X U_j$ such that $\pi_i(\omega) = x_i$ and $\pi_j(\omega) = x_j$.

Note that to each groupoid we can associate a simplicial space $X_\bullet$ where $X_n = X_1 \times_{X_0} \times \cdots \times_{X_0} X_1$ is the space of n-tuples of composable morphisms.

Definition 1.5. Let X, Y be topological groupoids. Then $F : X \to Y$ is an equivalence if it is essentially surjective and fully faithful. F is a local equivalence if F is an equivalence and for each $y \in Y_0$ there is a neighborhood U as in the figure below.

Remark 1.6.

- This notion of equivalence is not an equivalence relation. We end up with weird correspondence diagrams to make things work, which is reminiscent of some homotopy category stuff.

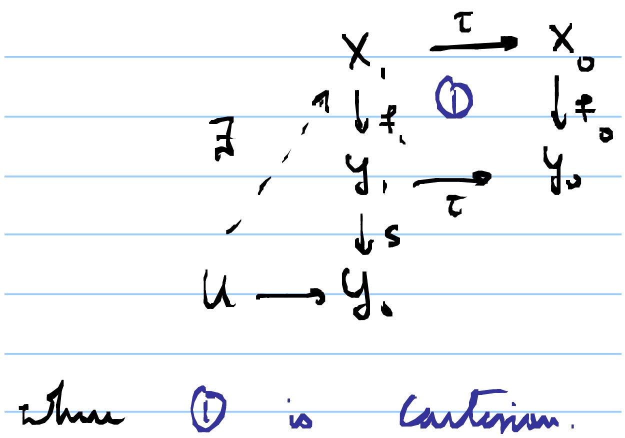

FIGURE 1. A local equivalence.

- The local equivalence basically enforces some notion of local lifting, which we don't get for equivalences because essentially surjective does not imply surjective. A groupoid defines a sheaf of groupoids on the site of spaces and local equivalences correspond to isomorphisms of sheaves.

Examples 1.7.

(1) For a refinement of covers, $U \to V$, there is a local equivalence $N_U \to N_V$.
(2) $G \to P \to X$ a principle bundle, then $P//G \to \; X \rightrightarrows X \;$ is a local equivalence.
(3) For a subgroup $H < G$, we have a local equivalence $P//H \to G/H//G$

Definition 1.8.

- A global quotient groupoid is one that is related via a zig-zag of local equivalences to a groupoid of the form $X//G$ for X Hausdorff and G a compact Lie group.
- A local quotient groupoid is one who admits a cover by open groupoids that are global quotients.

1.2. **Bundles over groupoids.** Before defining bundles and extensions of groupoids let us fix some notation. Let $X = (X_0, X_1)$ be a groupoid and suppose we have a bundle $P \to X_1$. We will denote by P_{f_i} the pull back of P along the map $X_n \to X_1$ given by

$$(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n) \mapsto (x_{i-1} \xrightarrow{f_i} x_i).$$

Similarly, if $(a \xrightarrow{f} b) \in X_1$ and $Q \to X_0$ is a bundle, then we have the pullbacks Q_a and Q_b on X_1.

Definition 1.9. For $\; X_1 \rightrightarrows X_0 \;$ a groupoid, a fiber bundle is a fiber bundle $P \to X_0$ together with a bundle isomorphism $t_f : P_a \to P_b$ for $f : a \to b \in X_1$

such that $t_{Id} = Id$ and satisfying a cocycle condition, so that

$$\begin{array}{ccc} P_a & \xrightarrow{t_f} & P_b \\ & {\scriptstyle t_{g\circ f}} \searrow \quad \swarrow {\scriptstyle t_g} & \\ & P_c & \end{array}$$

commutes on X_2.

For $p : P \to X$ a fiber bundle write $\Gamma(P)$ for the space of sections

$$\Gamma(P) = \Gamma(X, P) = \{s : X \to P \mid p \circ s = Id_X\}$$

topologized as a subspace of

$$X_0^{P_0} \times X_1^{P_1}.$$

Bundles behave well with respect to pull back and descent, so we have the following.

Proposition 1.10. *Let $F : X \to Y$ be a local equivalence and $P \to Y$ be a fiber bundle.*

- *Then there is a homeomorphism,*

$$\Gamma(X, F^*P) \cong \Gamma(Y, P).$$

- *The pullback functor*

$$F^* : \{\textit{Fiber bundles over } Y\} \to \{\textit{Fiber bundles over } X\}$$

  is an equivalence of categories.

Later we will realize the K-theory of a groupoid as the space of sections of a certain bundle of Hilbert spaces over the groupoid.

1.3. **Central extensions.**

To incorporate twists into K-theory we will need to use central extensions of groupoids.

Definition 1.11. A $U(1)$ central extension of $X = (X_0, X_1)$ is a $U(1)$-bundle L over X_1, together with an isomorphism of $U(1)$-bundles on X_2

$$\lambda_{g,f} : L_g \otimes L_f \to L_{g\circ f}$$

such that the following diagram of $U(1)$-bundles commutes on X_3

$$\begin{array}{ccccc} (L_h \otimes L_g) \otimes L_f & \longrightarrow & L_h \otimes (L_g \otimes L_f) & \longrightarrow & L_h \otimes L_{g\circ f} \\ & \searrow & & & \downarrow \\ & & L_{h\circ g} \otimes L_f & \longrightarrow & L_{h\circ g\circ f} \end{array}$$

If $L \to X_1$ is a central extension of X, then the pair $\widetilde{X} = (X_0, L)$ is a groupoid over X and the functor $\widetilde{X} \to X$ represents $\widetilde{X}$ as a central extension of X in the obvious way. This perspective will be useful when we discuss twisted K-theory.

Examples 1.12.

- For $X = *//G$, a central extension is just a central extension of G by $U(1)$.
- It is a fact that (up to isomorphism) $U(1)$-bundle gerbes over X are in bijection with central extensions of X. Also, to a $\mathbb{P}(H)$-bundle there is an associated $U(1)$-gerbe called the lifting gerbe. As an example, we define M_T. Consider $S^1 \times S^1 \to S^1 \times S^1$ given by $(z_1, z_2) \mapsto (z_1 z_2, z_2)$. Then we take the mapping torus for $B(S^1 \times S^1)$:

$$\mathbb{CP}^\infty = BS^1 \hookrightarrow BS^1 \times BS^1 \times [0,1]/\sim \xrightarrow{p_2 \times p_3} BS^1 \times S^1 \xrightarrow{DD} K(\mathbb{Z}, 3)$$

where DD classifies the Dixmier-Douady class (this class is the obstruction to lifting to a Hilbert bundle). Here, $DD(M_T)$ is not torsion.

Definition 1.13. Let X be a local quotient groupoid, then a twist is a pair (P, L) where $P \to X$ is a local equivalence and L is a central extension of P.

Proposition 1.14. *If (P, L) is a twist of a local quotient groupoid, then the groupoid $\widetilde{P} = (P_0, L)$ is a local quotient groupoid.*

Remark 1.15. We should really incorporate a $\mathbb{Z}/2$ grading into everything above, but for clarity we've neglected this. Further, the twists of a local quotient groupoid X form a monoidal category; we will not expand on this aspect in these notes.

Examples 1.16.

- Let X be a space and $P \to X$ a principal G-bundle. Further, let $\widetilde{G} \to G$ be a central extension, then $(P//\widetilde{G}, P//G)$ is a twisting of X.
- Suppose that G is a connected, compact Lie group, then we have the path-loop fibration

$$\Omega G \to PG \to G,$$

so we can regard PG as a principal bundle over G. Note that G acts on all spaces by conjugation. Let LG be the free loop space, then via the evaluation (at 1) map $LG \to G$ we have the semidirect product decomposition

$$LG \cong \Omega G \rtimes G.$$

The group LG acts on the fibration above by conjugation and the action of LG on G factors through the action of G on itself by conjugation through the evaluation map $LG \to G$. As a result we have a local equivalence of groupoids

$$PG//LG \to G//G.$$

Hence, a central extension $\widetilde{LG} \to LG$ defines a twisting of $G//G$.

2. Hilbert Bundles and K-theory

Using the language of Hilbert bundles we define twisted K-theory for local quotient groupoids.

Definition 2.1. A Hilbert bundle H on X is a fiber bundle with fiber a $\mathbb{Z}/2$-graded separable Hilbert space. H is universal if it contains any other Hilbert bundle as a summand. H is locally universal if for all open subgroupoids $X_U \hookrightarrow X$, $H|_{X_U}$ is universal.

Proposition 2.2.

- *Suppose X is a global quotient, $X = S//G$, then*
$$H = S \times L^2(G) \times \mathbb{C}l_1 \times \ell^2$$
is locally universal.
- *If Y is a local quotient groupoid, then there exists a locally universal Hilbert bundle H.*
- *The bundles above are unique up to contractible choices.*

Notice that on a space, Hilbert bundles are always trivial so that these notions are only interesting if points have automorphisms.

Using the language of the above proposition we obtain an interesting characterization of local quotient groupoids.

Proposition 2.3. *A local quotient groupoid is a global quotient groupoid if and only if its universal Hilbert bundle splits as a finite sum of finite dimensional bundles.*

Corollary 2.4. *Any gerbe with non-torsion DD-class is not a global quotient.*

In particular, M_T is not a global quotient. Also, $A/S^1 \times LG$ is not a global quotient groupoid as it fibers over M_T, but it is a local quotient groupoid.

2.1. **K-Theory.** Let X be a local quotient groupoid, H its locally universal Hilbert bundle. Then define

$$Fred^{(0)}(H) = \{A \in Fred(H) | A^2 + I \text{ is compact}\}.$$

This does half the job, namely this gives us even K-theory. Now we need to get odd K-theory. So let $A \in Fred(\mathbb{C}l_n \otimes H)$ for n odd. Let

$$\omega(A) := \epsilon_1 \cdots \epsilon_n \cdot A \quad n = -1 \bmod 4$$
$$\omega(A) := i^{-1}\epsilon_1 \cdots \epsilon_n \cdot A \quad n = 1 \bmod 4,$$

where the ϵ_i are generators for the Clifford algebra. Then define

$$Fred^{(n)}(H) \subset Fred^{(0)}(\mathbb{C}l_n \otimes H)$$

as odd operators that commute with $\mathbb{C}l_n$ and such that $\omega(A)$ has positive and negative essential spectrum.

Now define

$$\underline{k}(X)_n = \Gamma(X, Fred^{(0)}(H)) \quad n \text{ even}$$
$$\underline{k}(X)_n = \Gamma(X, Fred^{(1)}(H)) \quad n \text{ odd}$$

and

$$K^n(X) = \pi_0(\underline{k}(X)_n).$$

Theorem 2.5. *K^* is functorial, and local equivalent groupoids give isomorphic K-theories. We have a MV sequence for open subgroupoids and shriek maps for K-oriented maps.*

If we apply K^* to a space, then we recover the K-theory of the space. Similarly, $K^*(X \rtimes G)$ is the G-equivariant K-theory of X.

Now, what about the twists? Recall that a twist $\tau = (P, L)$ of a local quotient groupoid X defines a local quotient groupoid $\widetilde{P}$ with an action of a central $U(1)$ on its universal Hilbert bundle. So we define

$$^{\tau}K(X) := [K(^{\tau}X)]_{deg\ 1}$$

where the degree 1 is with respect to the action of the central $U(1)$ on the universal Hilbert bundle.

2.2. **The Kac numerator.** Recall our motivating theorem.

Theorem 2.6.

$$^{\tau-\sigma}R(S^1 \rtimes LG) \cong {}^{\tau}K(A/S^1 \rtimes LG)$$

From $e \to G$ we get a shriek map

$$ind : {}^{\tau-\sigma}R(S^1 \rtimes G) \to {}^{\tau}K(A/S^1 \rtimes G)$$

and by dualizing with respect to the $R(S^1)$-module structure we have

$$ind^* : {}^{\tau}K(A/S^1 \rtimes LG) \to Hom_{\mathbb{Z}}(^{\tau-\sigma}R(G); R(S^1))$$

Let H be an irreducible representation of $S^1 \rtimes LG$. Then

$$ind^*[H] = \sum_{\mu} \epsilon(u) q^{||\mu||^2/2} Tr(g) V_{\rho-\mu}$$

where the right hand side is the Kac numerator and μ ranges over the Weyl orbit of $\lambda + \rho$ where λ is the lowest weight of H, $\rho = (0, \frac{1}{2}\sum_{\alpha>0} \alpha, 0)$ is a particular weight and V_μ are representations of G such that $\mu - \alpha$ is not a weight for all roots α.

For $q = 1$ we can interpret the above formal character as a delocalized Chern character. See [**?**] for further discussion.

References

[1] D.S. Freed, M.J. Hopkins, C. Teleman, *Loop groups and twisted K-theory I*, arXiv: math.AT/0711.1906.

[2] D.S. Freed, M.J. Hopkins, C. Teleman, *Twisted K-theory and loop group representations*, arXiv: math.AT/0312155.