THE TWISTED EQUIVARIANT CHERN CHARACTER

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The talk will have three stages:

(1) the "delocalized" Chern character for equivariant K-theory;
(2) the Chern character for twisted K-theory;
(3) (= (1) + (2)) the twisted equivariant Chern character and its use in computing ${}^{\tau}K^*_G(G)$ (over $\mathbb{C}$).

We won't include any proofs but we will do a few computations so that you walk away recognizing that you can do concrete things with these ideas.[1] At the very least, it shows that the FHT story over the rationals is quite accessible.

1. The Equivariant Chern Character

Recall that the ordinary Chern character gives us a ring isomorphism (after rationalizing)

$$ch : K_{\mathbb{Q}} \xrightarrow{\cong} H_{\mathbb{Q}}.$$

This map is quite helpful in getting a quick impression of the K-theory of a space, although it misses much of the really interesting information, since it only lives over $\mathbb{Q}$. Ideally, we'd like a similar map for equivariant K-theory, but we will discover that things are more complicated (and hence more interesting). Before thinking about equivariant K-theory, however, let's remind ourselves how equivariant *cohomology* works.

Recall the definition: $H^*_G(X) := H^*(EG \times_G X)$. In other words, the G-equivariant cohomology of a G-space X is the ordinary cohomology of the homotopy quotient $X//G$.[2]

Let's do some examples just to remind ourselves how this works.

Example 1.1. Consider $G = S^1$. Then

$$H^*_{S^1}(pt) = H^1(BS^1) \cong H^1(\mathbb{CP}^\infty) \cong \mathbb{Z}[t],$$

where the generator lives in degree 2. More generally, for a torus $T = (S^1)^n$ then

$$H^*_T(pt) = \mathbb{Z}[t_1, \ldots, t_n].$$

[1]This write-up is a fairly accurate record of what I wanted to say during the talk. By no means does it adequately describe the twisted equivariant Chern character, and it's undoubtedly riddled with misunderstandings. I strongly encourage the reader to pick up the paper [2]. Nonetheless, this write-up should hopefully make it easier for the reader, when she does read [2], to understand the motivating ideas, to focus on the essential aspects of the proofs, and to have a few basic examples to bear in mind. If you notice mistakes in this text, please email me so that I can fix them.

[2]Perhaps this notation isn't ideal. Since we won't use it again, feel free to pass over it.

Example 1.2. Now if $G = SU(2)$ we can use the Serre spectral sequence for the fibration $SU(2) \hookrightarrow pt \to BSU(2)$ to see that

$$H^*_{SU(2)}(pt) = \mathbb{Z}[t]$$

where now t lives in degree 4. If you write down the E_2 page of the sequence, you'll see that the cohomology of $BSU(2)$ is completely determined by trying to eliminate all the interesting cohomology classes of $H^*(SU(2))$. It's fun to try $H^*_{SU(n)}(pt)$ ...

Remark 1.3. In general, we need to use power series rather than polynomial rings to define the Chern character. For finite dimensional spaces, this is irrelevant, but it's important in the general case, and justified in that we can equally well extract a ring from a graded ring by taking power series rather than the conventional direct sum.

If life were simple, equivariant K-theory would also arise from the Borel construction, i.e., we would have

$$\text{“}K_G(X)\text{”} = K(EG \times_G X)$$

and so the regular Chern character $ch : K_{\mathbb{Q}}(EG \times_G X) \to H_{\mathbb{Q}}(EG \times_G X)$ would give us the "equivariant Chern character."

As Mio told us, things are not so simple. Recall the completion theorem of Atiyah and Segal.

Theorem 1.4. *$K(BG)$ is the completion of $K_G(pt)$ at the augmentation ideal.*

In other words, equivariant K-theory sees more than just the homotopy quotient of X. Thus the Chern character, naively defined, would give us incomplete information. One way to interpret this is that we can't understand the equivariant K-theory just using the methods of homotopy theory.

Let G be a compact Lie group. Recall that in the actual definition,

$$K_G(pt) = Rep(G) =: R(G).$$

We have a ring isomomorphism

$$\begin{array}{ccc} R(G) \otimes \mathbb{C} & \overset{\cong}{\to} & \text{character ring} \\ \text{representation } V & \mapsto & \chi_V \text{ its character} \end{array} \tag{1}$$

using standard arguments from representation theory. (Namely, use the Haar measure to make a G-invariant inner product ...)

N.B.: We will always work over $\mathbb{C}$ from now on. So cohomology, K-theory, and the like will all be complex vector spaces.

Example 1.5. For $G = S^1$ the representation ring is $\mathbb{C}[t, t^{-1}]$, which follows from basic Fourier analysis: the irreducible characters are just the exponential functions e^{inx}, where $x \in [0, 2\pi)$ and $n \in \mathbb{Z}$.

Example 1.6. Here's another example you all know. Let $G = GL_n(\mathbb{C})$. Characters only depend on the conjugacy class of group elements, and Jordan canonical form insures that we understand conjugacy classes of matrices. We thus see that the character χ_V of a finite-dimensional representation V is a symmetric polynomial in the generalized eigenvalues of the conjugacy class of the matrix, so

$$R(G) \cong \mathbb{C}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]^{S_n}.$$

(I should actually discuss $U(n)$ here, but I thought the idea was clearer by speaking of all matrices.)

In general, for $T \subset G$ a maximal torus,

$$R(G) \cong R(T)^W,$$

where W is the Weyl group of G.

Those audience members who are fond of algebraic geometry may have asked themselves the following question: the equivariant cohomology and equivariant K-theory of a point are both commutative rings, so what do the associated affine schemes look like? Returning to our examples, we see that cohomology gives us the adjoint quotient of the (complexified) Lie algebra:

$$\mathit{Spec}\, H_T^*(pt) = \mathbb{G}_a \times \cdots \times \mathbb{G}_a \cong \mathfrak{t}_{\mathbb{C}}$$

and

$$\mathit{Spec}\, H_G^*(pt) \cong \mathfrak{t}_{\mathbb{C}}/W \cong \mathfrak{g}_{\mathbb{C}}/G_{\mathbb{C}},$$

where here we take the GIT quotient, not the stacky quotient. By contrast, the equivariant K-theory sees the adjoint quotient of the group (rather, the complexified group):

$$\mathit{Spec}\, K_T^*(pt) = \mathbb{G}_m \times \cdots \times \mathbb{G}_m \cong T_{\mathbb{C}}$$

and

$$\mathit{Spec}\, K_G^*(pt) \cong T_{\mathbb{C}}/W \cong G_{\mathbb{C}}/G_{\mathbb{C}}.$$

These are concrete, natural spaces to think about. Moreover, our Chern character needs to relate these two adjoint quotients, as a kind of exponential map …there's something cool to say here, but I'm not sure what it is. Toly will hopefully explain this more lucidly.

Now since $K_G(X)$ is an $R(G)$-module, $K_G(X)$ gives a quasicoherent sheaf $\mathscr{K}_G(X)$ on $\mathit{Spec}\, R(G)$. The crucial theorem, proved in [2], says that we *can* understand this sheaf using equivariant cohomology.

Theorem 1.7 (Untwisted)**.** *For a point $p \in \mathit{Spec}(R(G))$,*

$$K_G^*(X)_p^\vee \cong H_{Z(g)}^*(X^g)$$

where here $g \in G$ is associated to the conjugacy class of p, $Z(g)$ is the centralizer of g, X^g denotes the fixed point set of the cyclic group $\langle g \rangle$,[3] *and $K_p^\vee$ means the completion of the module at the point p.*

How do we interpret this theorem? It says that we can describe the local data of the sheaf $\mathscr{K}_G(X)$ using equivariant cohomology. (In fact, the proof of the theorem says we can see not just completions, but étale localizations.) In other words, point by point on $\mathit{Spec}\, R(G)$ we have a kind of Chern character,

[3] We're working in the topological setting, so the cyclic group $\langle g \rangle$ means the topological closure of the (algebraic) cyclic group generated by g.

expressing the K-theory in terms of cohomology. This is what people mean by the "globalized" or "delocalized" Chern character: equivariant K-theory defines a quasicoherent sheaf whose "stalks" (really, completions or étale stalks) are given by equivariant cohomology. Thus, this theorem lets us describe equivariant K-theory (at least over $\mathbb{C}$) using the methods of homotopy theory.

Remark 1.8. One has to be careful about g – it needs to be *normal* – and you should look at the paper for a discussion of this issue. I'll mention it again later.

1.1. **Applying the theorem.** Let's do an example. Take S^2 with S^1 acting on it by rotation around the "z-axis" (namely, the line passing through the north and south poles, when we view S^2 as living in $\mathbb{R}^3$). Recall that

$$R(S^1) = \mathbb{C}[t, t^{-1}] \Longrightarrow Spec\, R(S^1) = \mathbb{G}_m(\mathbb{C}) \cong \mathbb{C}^\times.$$

Let's apply the theorem. There are two kinds of group elements to consider. First, let $g = e^{i\theta}$ for $\theta \neq 0$. The fixed points are exactly the poles: $X^g = \{N, S\}$; the centralizer is everything since the group is abelian: $Z(g) = S^1$. So the equivariant cohomology is

$$H^*_{S^1}(N \sqcup S) \cong \mathbb{C}[[t_N]] \oplus \mathbb{C}[[t_S]].$$

Notice that we've switched to using power series, which we should've done all along. The second kind of group element is the identity $g = 1$, in which case $X^g = X$. To compute the equivariant cohomology, we'll use Mayer-Vietoris. We choose a really simple equivariant cover: $U_S = S^2 - N$ and $U_N = S^2 - S$. Notice these charts play well with the S^1 action. The intersection is $U_S \cap U_N = S^2 - \{N, S\}$. We compute that

$$H^*_{S^1}(U_S) = \mathbb{C}[[t_S]], \quad H^*_{S^1}(U_N) \cong (C)[[t_N]], \quad H^*_{S^1}(U_S \cap U_N) \cong \mathbb{C},$$

since U_S deformation retracts equivariantly onto the south pole S (and likewise for U_N) and the intersection has a free action of S^1 (so that the homotopy quotient is the honest quotient, which is an interval, and hence has no interesting ordinary cohomology). Plugging into the Mayer-Vietoris long exact sequence, we see that $H^*_{S^1}(S^2)$ is the kernel of the map

$$\mathbb{C}[[t_N]] \oplus \mathbb{C}[[t_S]] \to \mathbb{C}$$

that reads off the constant parts of the respective power series.

So we've made the relevant equivariant cohomology computations. What does this tell us about the equivariant K-theory? Consider our K-theory sheaf $\mathscr{K}_{S^1}(S^2)$ over the subspace $S^1 \subset Spec\, R(S^1) = \mathbb{C}^\times$. What we've shown is that above the non-identity elements $e^{i\theta}$, our sheaf looks like the germ of two independent lines (corresponding to $\mathbb{C}[[t_N]] \oplus \mathbb{C}[[t_S]]$). Over the identity element, these two lines are glued together. All together, our sheaf looks like we took the trivial double cover over S^1 and pinched it together over the identity.

Exercise 1.9. *What happens if S^1 acts on S^2 by rotating at twice the speed? (This means that -1 acts trivially.) How does the naive equivariant Chern character compare to the delocalized Chern character?*

Remark 1.10. This exercise was suggested by Constantin, and he hinted that with examples like this, we can start to see the need for the delocalized story.

2. The Twisted Chern Character

Yesterday we began focusing on twists of K-theory of the form $(\tau, \epsilon) \in H^3(X;\mathbb{Z}) \oplus H^1(X;\mathbb{Z}/2)$. For simplicity (and due to my ignorance), we'll restrict from hereon to the case where $\epsilon = 0$. (Eventually this will force us to focus on simply-connected groups G.)

Before I launch into discussing twisted cohomology, let me state the main theorem. It's what you would hope for.

Theorem 2.1. *There is a functorial twisted Chern character*

$$^{\tau}ch : {}^{\tau}K(X,\mathbb{C}) \to {}^{\tau}H(X,\mathbb{C})$$

which is a module isomorphism over

$$ch : K(X,\mathbb{C}) \to H(X,\mathbb{C}).$$

What's great about this theorem is that there is a spectral sequence to compute ${}^{\tau}H(X;\mathbb{C})$, so that the twisted cohomology is (relatively) easy to compute. By contrast, homotopical methods don't directly access twisted K-theory very easily.

Usually twisted cohomology means cohomology with values in a local system, but here our twist lives in H^3.[4] In [2], Freed, Hopkins, and Teleman work in rational homotopy theory to define twisted cohomology (over $\mathbb{Q}$ in fact), but my guess is that those methods aren't familiar to everybody. Instead, I'll follow Atiyah and Segal in [1] and give a de Rham model, which is a bit easier to think about and to compute with. Pick a cocycle $\eta \in \Omega^3(X)$ representing τ. Then define

$$\begin{array}{rccc} D = D_\eta = d - \eta : & \Omega^{even}(X) & \to & \Omega^{odd}(X) \\ & \omega & \mapsto & d\omega - \eta \wedge \omega \end{array}$$

and vice versa from odd to even. Notice that

$$D^2 = d^2 - \eta d - d\eta + \eta^2 = 0,$$

using the Leibniz rule and the fact that η is odd. Hence we can define

$$H^0_\eta(X) = \text{even cohomology}, \quad H^1_\eta(X) = \text{odd cohomology}$$

of this 2-periodic complex. We will define ${}^{\tau}H(X)$ to be the cohomology we computed using η.

[4]This really isn't that weird. After all, rational K-theory is isomorphic to a 2-periodic version of rational cohomology. So using the twist τ to construct a bundle of K-theory spectra will automatically lead to a bundle made out of the associated Eilenberg-MacLane spectra. Maybe more simply, we see that the usual Chern character induces an isomorphism between rational K-theory and periodic rational cohomology locally in X, and we're just gluing it together.

Let's see how the twisted cohomology changes when we take a different representative for τ. Take η' another representative, where $\eta - \eta' = d\xi$. Think of this like a "change of gauge." We get a formula

$$D_{\eta'} e^{\xi} = e^{\xi} D_{\eta}.$$

This gives an isomorphism

$$H_{\eta}(X) \cong H_{\eta'}(X).$$

So we see that $H^2(X; \mathbb{C})$ acts on $H_\eta(X)$ via "gauge change" by an element $\xi \in \Omega^2$. This action is just an avatar of the action of line bundles on twisted K-theory

$$Pic(X) \times {}^{\tau}K(X) \to {}^{\tau}K(X)$$
$$(\mathcal{L}, V) \mapsto \mathcal{L} \otimes V$$

that we discussed yesterday. We define ${}^{\tau}H$ as H_η, but we do need to make a choice for η and we do need to remember this action of H^2 to get a well-behaved theory.

Now let's describe this spectral sequence that's so useful. There is a natural filtration on Ω^*, namely, $F^p\Omega^* = \oplus_{q \geq p} \Omega^q$, and our twisted differential D preserves it. Thus we get a spectral sequence

$$E_2^{pq}(X) \Rightarrow {}^{\tau}H^{p+q}(X)$$

where E_2^{pq} is $H^p(X)$ for q even and 0 for q odd, the E_2 differential d_2 vanishes everywhere, and the E_3 differential is cupping with τ: $d_3 = \tau \cup -$.

Example 2.2. Take $G = SU(2)$. Let's compute the twisted cohomology of G. Since $SU(2)$ is the 3-sphere, we know

$$H^*(SU(2)) \cong \mathbb{C}[v]$$

where v is degree 3 and $v^2 = 0$ (because it's odd). Hence τ is some scalar multiple of v. For $\tau \neq 0$, we discover

$${}^{\tau}H^*(SU(2)) = 0$$

which follows quickly after looking at the spectral sequence: everything gets hit by cupping with v. Can you now compute the twisted equivariant cohomology of G acting on itself by conjugation?

3. The Twisted Equivariant Chern Character

We will combine our work in the last two sections. I just want to state the main theorem and then apply it to an example, but we need to pin down some notation first.

- Let G be a compact Lie group and $G_{\mathbb{C}}$ its complexification. (For concreteness, think of $SU(n)$ and $SL_n(\mathbb{C})$.)
- The element $g \in G_{\mathbb{C}}$ is "normal" (i.e. it commutes with hermitian adjoint). This implies

  $$\langle g \rangle := G \cap \langle g \rangle_{\mathbb{C}}$$

  is cyclic and

  $$Z(g) := G \cap Z_{\mathbb{C}}(g)$$

is the centralizer.

- X^g denotes the fixed point set of $\langle g \rangle$.
- ${}^\tau K_G(X)^\vee_q$ denotes the formal completion of the twisted equivariant K-theory at the conjugacy class $q \in Spec\, R(g)$.
- ${}^\tau\mathcal{L}(g)$ on $X^g =: Y$ denotes a flat $Z(g)$-equivariant line bundle.

We construct the line bundle ${}^\tau\mathcal{L}(g)$ as follows. Pick a projective Hilbert bundle

$$\pi : \mathbb{P}_Y \to Y$$

associated to the twisting τ that lifts the $Z(g)$ action on Y projectively. At each point $y \in Y$, $\langle g \rangle \subset Z(g)$ acts projectively on the fiber $\pi^{-1}(y)$ and so we get a central extension, $\widetilde{\langle g \rangle}_y$, coming from the lifting:

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{C}^\times & \to & GL & \to & PGL & \to & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \to & \mathbb{C}^\times & \to & \widetilde{\langle g \rangle} & \to & \langle g \rangle & \to & 1 \end{array} \tag{2}$$

This give a principal $\mathbb{C}^\times$-bundle ${}^\tau\mathcal{L}$ on $Y \times \langle g \rangle_\mathbb{C}$. We define ${}^\tau\mathcal{L}(g)$ to be the restriction of this line bundle to $Y \times g$.

We now have a line bundle, but it is not yet clear that it is flat. We see this as follows. Observe that $\langle g \rangle$ is topologically cyclic (i.e., a torus cross a finite cyclic group), so all central extensions are trivial. There are, however, many ways to trivialize the extension. Given two such splittings, they differ by a homomorphism

$$\rho : \langle g \rangle_\mathbb{C} \to \mathbb{C}^\times,$$

and the holonomy given by following a splitting around a closed loop in $\langle g \rangle$ is $\rho(g)$.

Constantine: Flat line bundles are determined by holonomy, so for each loop we want to produce a complex number. A twisting is a 3-cocycle, and a loop is a 1-cycle, and contracting the 3-cocycle of the twisting with the 1-cycle of the loop and a 2-cycle coming from exponentiating a Lie algebra element produces a complex number, which happens to be invertible, giving our flat connection.

The main theorem of [2] is the twisted version of the delocalized Chern character we discussed earlier.

Theorem 3.1.

$${}^\tau K^*_G(X)^\vee_g \xrightarrow{\cong} {}^\tau H^*_{Z(g)}(X^g, {}^\tau\mathcal{L}(g))$$

Again, we have described the local data of a sheaf ${}^\tau\mathscr{K}_G(X)$ over $Spec\, R(G)$ using twisted equivariant cohomology. This is our "delocalized" twisted equivariant Chern character. Since we have a spectral sequence for computing twisted equivariant cohomology, we can try to use homotopy theory to access the K-theory.

Here's a crucial fact shown in [2]: for a group G with $\pi_1 G$ free, ${}^\tau\mathcal{K}_G(G)$ is a skyshaper sheaf for G acting on itself by conjugation. The support is a finite group living inside the maximal torus $T \subset G$.[5] This means that knowing the completions is enough! We've reduced the problem of computing the twisted equivariant K-theory of a group acting on itself by conjugation down to the problem of computing twisted cohomology, which is easy since we have the spectral sequence.

Example 3.2. Now let's do the example $SU(2)$. Recall

$$Rep(SU(2)) = \mathbb{C}[t, t^{-1}]^{S_2}.$$

There is a map

$$\pi : Spec(Rep(SU(2)) \cong \mathbb{G}_m/S_2 \cong SL_2\mathbb{C}/SL_2\mathbb{C} \to \mathbb{A}^1$$

which takes the trace of a conjugacy class of a matrix. Explicitly,

$$\pi : \left[\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\right] \mapsto \lambda + \lambda^{-1}.$$

Now, restricting our attention to the maximal compact,

$$SU(2) \hookrightarrow SL_2\mathbb{C}$$

and the map takes $diag(e^{i\theta}, e^{-i\theta}) \mapsto e^{i\theta} + e^{-i\theta} \in [-2, 2] \subset \mathbb{A}^1$.

For $g = diag(e^{i\theta}, e^{-i\theta})$ with nonzero θ,

$$\langle g \rangle_{\mathbb{C}} = T_{\mathbb{C}},$$

or a cyclic group if $e^{i\theta}$ is a root of unity. We also compute

$$Z(g) = T \text{ and } SU(2)^{\langle g \rangle} = T.$$

When $\theta = 0$ or π, then $Z(g) = G$ and $X^g = G$.

It turns out (as Dan H-L will show)

$$H^3_G(G; \mathbb{Z}) \cong \mathbb{Z},$$

so we know the space of twistings.

Let's focus on the cases where $\tau \neq 0$ and $g \neq \pm 1$. One can show

$${}^{\tau}\mathcal{L}(g) \to S^1 = SU(2)^{\langle g \rangle}$$

is a line bundle over the circle with holonomy $\lambda^{2\tau}$. This implies

$${}^{\tau}H_{Z(g)}(S^1; {}^{\tau}\mathcal{L}(g)) = 0$$

if $\lambda^{2\tau} \neq 1$. So to be interesting, we should choose λ to be a root of unity,

$$\lambda_k = e^{i\pi k}/\tau, \quad k = 1, \ldots, \tau - 1.$$

[5]I didn't explain what the support was in the talk, but it's easy to describe. We can pull back τ from $H^3_G(G)$ to $H^3_T(T)$ along the inclusion of the maximal torus. Notice that $H^1(T) \otimes H^2(BT)$ is a summand of $H^3_T(T)$ by using the Serre spectral sequence. Consider the piece of τ living in this summand, which we identify with $\mathrm{Hom}_{\mathbb{Z}}(H_2(BT), H^1(T))$. We focus our attention on *regular* twists τ, which means the restriction of τ to this summand is full rank as a $\mathbb{Z}$-linear map. Since these groups $H_2(BT)$ and $H^1(T)$ are integral lattices inside the associated real co/homology, the map τ descends to a map from the torus T to its dual torus $T^\vee$. We're using here the natural interpretation of the dual torus $T^\vee$ as the space of complex line bundles on the torus T, and the fact that $H_2(BT) = H_1(T)$ describes the 1-parameter subgroups of the torus T. Finally, the support of the twisted equivariant K-theory, viewed as a sheaf on T, is the kernel of the map defined by τ.

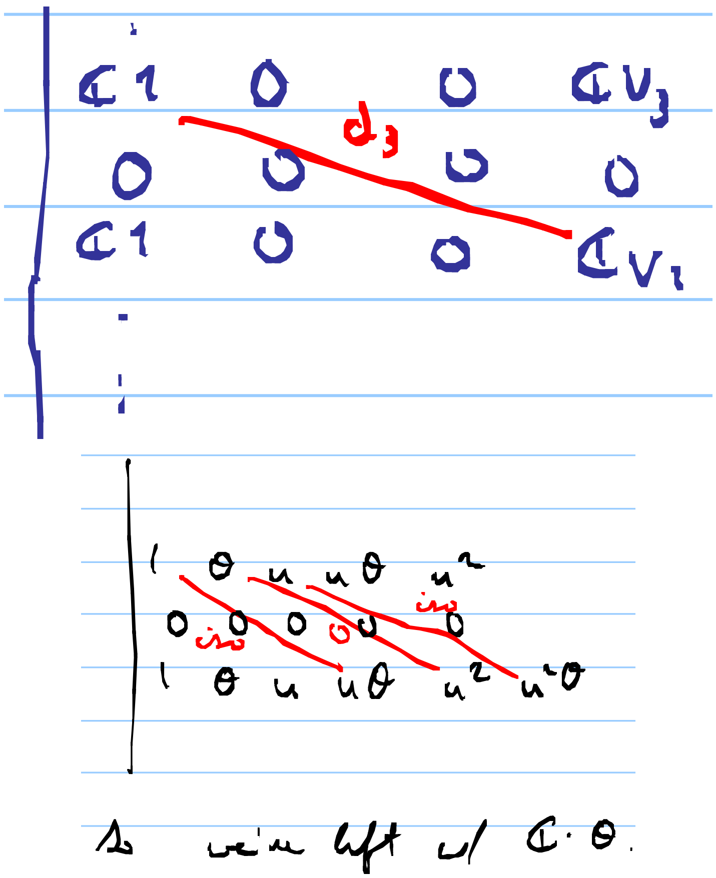

Figure 1. The spectral sequences.

For the circle $T = S^1$,

$$H_T^*(T) \cong H^*(BT) \otimes H^*(T) \cong \mathbb{C}[[u, \theta]]/\theta^2$$

where u is the generator for $BT = \mathbb{CP}^\infty$ and θ is the generator for S^1. (The 2 is coming from the restriction map of $H_G^3(G)$ to $H_T^3(T)$. Recall that the Weyl group is S_2.)

Now let's look at the spectral sequence for computing twisted cohomology. The d_3 differential is cupping with $2\tau \cdot u\theta$, so the sequence collapses at the E_4 page and we see that the twisted cohomology is precisely $C[\theta]/\theta^2$.

4. Concluding remark

The astute reader has undoubtedly noticed that I've never actually described any of the Chern character maps mentioned so far. We've only discussed the nature of the target (e.g., twisted cohomology) and used the main theorems to construct the sheaf. Unfortunately, given a class in twisted equivariant K-theory, I do not have a feel for what its Chern character is! I encourage the reader to delve more deeply into [2] to figure this issue out.

References

[1] Michael Atiyah and Graeme Segal, "Twisted K-theory and Cohomology," math.KT/0510674

[2] Daniel Freed, Michael Hopkins, and Constantin Teleman, "Twisted equivariant K-theory with complex coefficients," math. AT/0206257

[3] Ioanid Rosu and Allen Knutson, "Equivariant K-theory and Equivariant Cohomology," math/9912088