1. Equivariant Twisted $K$-Theory, Mio Alter, University of Texas

First we’ll go through equivariant $K$-theory via vector bundles and $C^*$-algebras, then review the completion theorem, then do a twisted equivariant example.

1.1. Equivariant $K$-theory via vector bundles.

**Definition 1.1.** If $X$ is compact space and $G$ is a compact Lie group, an equivariant vector bundle $\pi: E \to X$ is a vector bundle such that $\pi$ is equivariant and $E_x \mapsto E_{gx}$ is linear for each $x \in X$.

**Proposition 1.2.** If $G$ acts freely on $X$, $\text{Vect}_G(X) \cong \text{Vect}(X/G)$.

We see this as

\[
\begin{array}{c}
E \quad \mapsto \\
\downarrow \quad \downarrow \\
X \quad X/G
\end{array}
\]

**Proposition 1.3.** If $G$ acts trivially on $X$ and $F \to X$ is an equivariant vector bundle, then

\[
F \cong \bigoplus_{i=1}^n V_i \otimes E_i
\]

where $V_i = X \times V_i$, $V_i$ an irreducible representation of $G$ and $E_i \cong \text{Hom}_G(V_i, F)$ is an equivariant vector bundle with trivial $G$-action.

**Definition 1.4.** $K^*_G(X)$ is the $K$-theory of $G$-equivariant vector bundles over $X$, i.e. $K^*_G(X) = K^*(\text{Vect}_G(X))$.

**Proposition 1.5.** For $X$ a trivial $G$-space, $K^*_G(X) \cong R(G) \otimes K^*(X)$, and $R(G) \cong K_G(\text{pt})$.

Example: let $H \subset G$ be a closed subgroup, $E \to G/H$ a $G$-vector bundle. Then $E_H$ is an $H$-module; we show that $E$ is completely determined by $E_H$. Let $G \times_H E_H \to G/H$ be the bundle with $G$-action,

\[
g \cdot [\tilde{g}, \xi] = [g\tilde{g}, \xi]
\]

There is an obvious bundle map

\[
\begin{array}{ccc}
G \times_H E_H & \xrightarrow{\alpha} & E \\
\downarrow \alpha & & \downarrow \\
G/H & \xrightarrow{g \cdot [\tilde{g}, \xi]} & G/H \\
\end{array}
\]

For $\eta \in E_{gH}$, define $\beta: E \to G \times_G E_H$ by

\[
\beta(\eta) = [g, g^{-1}\eta]
\]
It is clear that if we choose a different coset representative, we get the same thing, so $\beta$ is well-defined. It is immediate that $\beta$ is a 2-sided inverse to $\alpha$ so $E \cong G \times_H E_H$.

Thus, the category of $G$-vector bundles on $G/H$ is equivalent to the category of $H$-modules, $$\text{Vect}_G(G/H) \cong \text{Vect}_H(pt)$$

It follows that $$K^*_G(G/H) = K^*_H(pt) = \begin{cases} R(H) & * = 0 \\ 0 & * = 1 \end{cases}$$

Remark: There is a localization theorem for equivariant $K$-theory which is analogous to the localization theorem for equivariant cohomology.

One might also start with the Borel construction for equivariant $K$-theory and take the equivariant $K$-theory of a space to be just the ordinary $K$-theory of the Borel quotient $$X_G \cong EG \times_G X.$$ However, the equivariant bundle picture is much richer. In fact,

**Theorem 1.6** (Atiyah-Segal Completion).

$$\hat{K}^*_G(X) \cong K^*(X_G).$$

where $I$ is the augmentation ideal (the kernel of the augmentation map $\epsilon : R(G) \to \mathbb{Z}$ given by evaluating at $1 \in G$), that is, the ideal generated by virtual representations of virtual dimension zero.

**1.2. Equivariant $K$-theory via $C^*$-algebras.** Recall that a $C^*$-algebra is a Banach algebra with a $\mathbb{C}$-linear involution. All $C^*$-algebras have embeddings into $B(H)$ for some Hilbert space $H$. An example of a $C^*$-algebra is the ring of continuous functions $C(X, \mathbb{C})$, where $X$ is a compact space.

**Definition 1.7.** For $G$ a compact Lie group and $A$ a $C^*$-algebra, a continuous action of $G$ on $A$ is a map $\alpha : G \to \text{Aut}(A)$ where for a sequence $g_n \to g$, $\alpha(g_n)(a) \to \alpha(g)(a)$ for all $a \in A$.

**Definition 1.8.** A $(G,A,\alpha)$-module is an $A$-module in which $G$ acts compatibly with the $A$-action.

**Definition 1.9.** $K^*_G(A)$ is the Groethendieck group of isomorphism classes of projective $(G,A,\alpha)$-modules.

**Definition 1.10.** The crossed product $G \times_\alpha A$ is a completion of the twisted $C^*$-algebra, $C(G) \otimes A$ where the product (convolution) and involution are defined in terms of $\alpha$.

**Theorem 1.11.** $K^*_0(A) \cong K_0(G \times_\alpha A)$
Comment: so for $C^*$-algebras, unlike for spaces as seen by the Completion theorem, the equivariant $K$-theory of a $G$-algebra is the ordinary $K$-theory of an associated ordinary algebra.

Also, note that if your $C^*$-algebra happens to be the algebra of functions on a compact Hausdorff space $X$, the (equivariant) $K$-theory of the space agrees with the (equivariant) $K$-theory of the $C^*$-algebra,

$$K^*_G(X) \cong K^*_G(C(X))$$

1.3. Atiyah-Segal Construction of Twisted Equivariant $K$-Theory.

**Proposition 1.12.** Classes in $H^3(X; \mathbb{Z})$ are in bijection with projective Hilbert bundles $P \to X$ up to isomorphism.

**Proof.** To go from a bundle to a class, consider a cover of $X$ on which $P_\alpha \cong \mathbb{P}(E_\alpha)$, $E_\alpha$ a Hilbert bundle over $U_\alpha$. Then there are transition functions

$$g_{\alpha\beta} : P_\alpha \to P_\beta,$$

which we can lift to

$$\tilde{g}_{\alpha\beta} : E_\alpha \to E_\beta,$$

but there is a cocycle condition on these, namely

$$\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} : X_{\alpha\beta\gamma} \to U(1),$$

and this defines a class $\eta \in H^2(X; U(1)) \cong H^3(X; \mathbb{Z})$ where the isomorphism follows from the exponential sequence of sheaves $0 \to sh(\mathbb{Z}) \to sh(\mathbb{R}) \to sh(U(1)) \to 0$ of continuous $\mathbb{Z}, \mathbb{R},$ and $U(1)$-valued functions and $H^i(X; \mathbb{R}) = 0$ for $i > 0$ since $sh(\mathbb{R})$ has partitions of unity.

Less concretely (but for a quick proof of both directions in the theorem) $PU(H)$ is a $K(\mathbb{Z}, 2)$, so $BPU(H)$ is a $K(\mathbb{Z}, 3)$, and then we have that

$$H^3(X; \mathbb{Z}) \cong [X, BPU(H)],$$

which in turn is projective Hilbert bundles, up to isomorphism. □

Now given $\tau \in H^3(X; \mathbb{Z})$ Let $\tau \mathbb{P} \to X$ be the corresponding projective Hilbert bundle. Let $Fred(\tau \mathbb{P}) \to X$ be the associated bundle with fiber $Fred(H)$. Similarly let $\tau \mathbb{K} \to X$ be the associated bundle with fiber compact operators on $H$.

Now we define twisted $K$-theory as

$$\tau K^0(X) \cong \pi_0(\Gamma(X; Fred(\tau \mathbb{P})))$$

$$\tau K^0(X) \cong \tilde{K}_0(\Gamma(X; \tau \mathbb{K} \oplus I \cdot \mathbb{C}))$$

where these are sections that are a compact operator plus the identity.

We remark that the automorphisms of these projective Hilbert bundles come from tensoring with a line bundle.
1.4. Twisted Equivariant Story.

**Theorem 1.13** (Atiyah-Segal). There is a bijection between $G$-equivariant projective Hilbert bundles over $X$ and projective Hilbert bundles over $X_G$, the homotopy quotient.

So $G$-equivariant twists are in bijection with $H^3_G(X; \mathbb{Z})$.

Then we define

$$\tau K^0_G(X) := \pi_0(\Gamma(X; Fred(\mathcal{T}P)))$$

We can also take as our definition $\tilde{K}_0$ of the category of continuous projective $(G, \Gamma(X; \tau K \oplus I \cdot \mathbb{C}))-\text{modules}$. This is also equal to

$$\tilde{K}_0(G \times_\alpha \Gamma(X; \tau K \oplus I \cdot \mathbb{C})).$$

1.5. A Computation. $H^1(X; \{G-\text{Line Bundles}\}) \cong H^1_G(X; \{\text{Line Bundles}\}) \cong H^3_G(X; \mathbb{Z})$

Let’s consider the example where $X = U(1)$ acts on itself by conjugation (i.e. trivially); we still get interesting vector bundles, since the fibers are representations of $U(1)$. Take the open cover $U = U(1) \setminus \{1\}$ and $V = U(1) \setminus \{-1\}$, let $L$ be the defining representation of $U(1)$ and fix a twisting $\tau \in H^3_G(U(1), \mathbb{Z}) \cong \mathbb{Z}$ corresponding to $n$ under this isomorphism. A representative of the Cech 1-cocycle corresponding to this twist is the equivariant line bundle $L^n$ on $U \cap V^+$ and $1$ on $U \cap V^-$. We have the Mayer-Vietoris sequence

$$\begin{array}{c}
\tau K^0_{U(1)}(U) \\
\uparrow
\end{array}
\rightarrow
\tau K^0_{U(1)}(U) \oplus \tau K^0_{U(1)}(V) \\
\rightarrow
\tau K^0_{U(1)}(U \cap V)
$$

(2) but

$$\begin{array}{c}
\tau K^1_{U(1)}(U) \cong \tau K^1_{U(1)}(pt) = 0,
\end{array}$$

so we get a sequence

$$0 \rightarrow \tau K^0_{U(1)}(U(1)) \rightarrow \mathbb{Z}[L^\pm]^2 \rightarrow \mathbb{Z}[L^\pm]^2 \rightarrow \tau K^1_{U(1)}(U) \rightarrow 0,$$

where the middle map is given by $(x, y) \mapsto (x - y, x - y^n)$ since restriction from $U$ is straight restriction, but when we restrict from $V$, we tensor with the line bundle on the intersection. This shows

$$\tau K^1_{U(1)}(U(1)) \cong \mathbb{Z}[L^\pm]/ < 1 - L^n >,$$

and $0$ otherwise.

Remark: when we add $\mathbb{C} \cdot I$ in the above, we need to take reduced $K$-theory. It’s like adding a disjoint basepoint.