THE HOMOTOPY GROUPS OF *tmf* AND OF ITS LOCALIZATIONS

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In this survey, I present a compilation of the homotopy groups of *tmf* and of its various localizations. This work was an exercise in collecting the diffuse knowledge from my mathematical surroundings.

1. The homotopy of *tmf*

The spectrum *tmf* is connective, which means that the ring $\pi_n(tmf)$ is zero for $n < 0$. Vaguely speaking, its homotopy ring $\pi_*(tmf)$ is an amalgam of $MF_* = \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - (12)^3\Delta)$, the ring of classical modular forms, and part of $\pi_*(\mathbb{S})$, the ring of stable homotopy groups of spheres. More concretely, there are two ring homomorphisms

$$\pi_*(\mathbb{S}) \longrightarrow \pi_*(tmf) \longrightarrow MF_* \tag{1}$$

The map from $\pi_*(\mathbb{S})$ captures information about the torsion part of $\pi_*(tmf)$, while the second map is almost an isomorphism between the non-torsion part of $\pi_*(tmf)$ and MF_*.

The first map in (1) is the Hurewicz homomorphism. Since *tmf* is a ring spectrum, it admits a unit map from the sphere spectrum $\mathbb{S}$. This induces a map in homotopy $\pi_*(\mathbb{S}) \to \pi_*(tmf)$, which is an isomorphism on $\pi_0, \pi_1, \ldots, \pi_6$. The only torsion in $\pi_*(tmf)$ is 2-torsion and 3-torsion; it is at those primes that *tmf* resembles the sphere spectrum. The 3-primary part of the Hurewicz image is 72-periodic and is given by

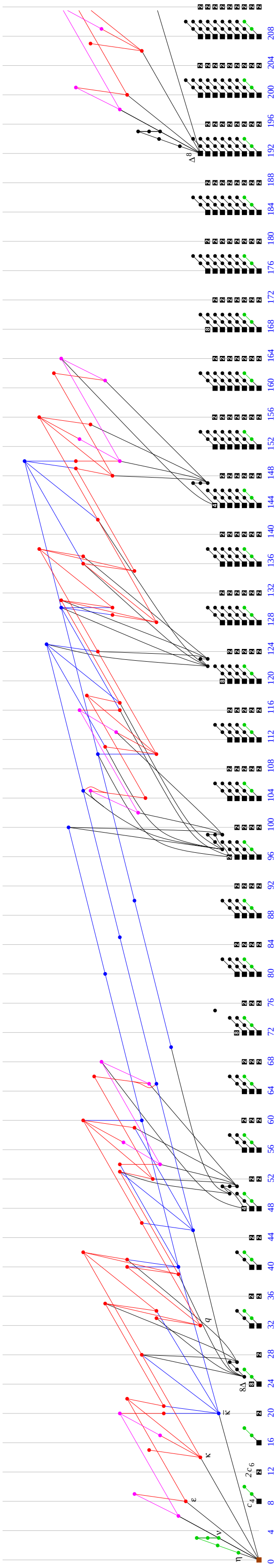
$$\mathrm{im}\Big(\pi_*(\mathbb{S}) \to \pi_*(tmf)\Big)_{(3)} = \mathbb{Z}_{(3)} \oplus \alpha\,\mathbb{Z}/3\mathbb{Z} \oplus \bigoplus_{k\geq 0} \Delta^{3k}\{\beta, \alpha\beta, \beta^2, \beta^3, \beta^4/\alpha, \beta^4\}\,\mathbb{Z}/3\mathbb{Z}, \tag{2}$$

where α has degree 3, and $\beta = \langle \alpha, \alpha, \alpha \rangle$ has degree 10. The Hurewicz image contains most but not all the 3-torsion of $\pi_*(tmf)$: the classes in dimensions $27 + 72k$ and $75 + 72k$ for $k \geq 0$ are not hit by elements of $\pi_*(\mathbb{S})$. The 2-torsion of $\mathrm{im}(\pi_*(\mathbb{S}) \to \pi_*(tmf))$ is substantially more complicated. It exhibits very rich patterns including two distinct periodicity phenomena. The first one is a periodicity by $c_4 \in \pi_8(tmf)$, which corresponds to v_1^4; the second one is a periodicity by $\Delta^8 \in \pi_{192}(tmf)$, which corresponds to v_2^{32}.

The second map in (1) is the boundary homomorphism of the elliptic spectral sequence. Under that map, a class in $\pi_n(tmf)$ maps to a modular form of weight $n/2$ (and to zero if n is odd). That map is an isomorphism after inverting the primes 2 and 3, which means that both its kernel and its cokernel are 2- and 3- torsion. Its cokernel can be described explicitly

$$\mathrm{coker}\Big(\pi_n(tmf) \to MF_{\frac{n}{2}}\Big) = \begin{cases} \mathbb{Z}\Big/\tfrac{24}{\gcd(k,24)}\,\mathbb{Z}, & \text{if} \quad n = 24k \\ (\mathbb{Z}/2\mathbb{Z})^{\lceil \frac{n-8}{24} \rceil} & \text{if} \quad n \equiv 4 \pmod 8 \\ 0 & \text{otherwise,} \end{cases}$$

where the first cyclic group is generated by Δ^k and the second group is generated by $\Delta^a c_4^b c_6$ for integers a and b satisfying $24a + 8b + 12 = n$. Its kernel agrees with the torsion in $\pi_*(tmf)$ and is much more complicated – it resembles the stable homotopy groups of spheres. The 3-primary component of the kernel is at most $\mathbb{Z}/3\mathbb{Z}$ in any given degree. The 2-primary component is a direct sum of $(\mathbb{Z}/2\mathbb{Z})^\ell$ for some ℓ (corresponding to v_1-periodic elements) with a group isomorphic to $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/8\mathbb{Z}$, or $(\mathbb{Z}/2\mathbb{Z})^2$ (corresponding to v_2-periodic elements).



The homotopy groups of *tmf* at the prime 2

The picture on the left of this page represents the homotopy ring of *tmf* at the prime 2. The vertical direction has no meaning. Bullets represent $\mathbb{Z}/2\mathbb{Z}$'s while squares represent $\mathbb{Z}_{(2)}$'s. A chain of n bullets connected by vertical lines represent a $\mathbb{Z}/2^n\mathbb{Z}$.

The bullets are named by the classes in $\pi_*(\mathbb{S})$ of which they are the image (η, ν, ε, $\bar{\kappa}$, κ, q are standard names), while the squares are named after their images in MF_*. The slanted lines represent multiplication by η, ν, ε, κ, and $\bar{\kappa}$.

The top part of the diagram is 192-periodic with polynomial generator Δ^8. We have colored the image of the Hurewicz homomorphism [conjectured by Mark Mahowald] as follows: the v_1-periodic classes are in green, and the v_2-periodic classes are in pink, red, and blue, depending on their periodicity. The green classes are v_1^4-periodic in the sphere, and, except for ν, they remain periodic in *tmf* via the identification $c_4 = v_1^4$. The v_2^8-periodic classes are pink, the v_2^{16}-periodic red, and the v_2^{32}-periodic blue. They remain periodic in *tmf* via the identification $\Delta^8 = v_2^{32}$.

The tiny white numbers written in the squares indicate the size of $\mathrm{coker}(\pi_*(tmf) \to MF_*)_{(2)}$. The $\mathbb{Z}_{(2)}$-algebra $\pi_*(tmf)_{(2)}$ is finitely generated, with generators

degree:	1	3	8	8	12	14	20	24	25	27	32	32
name:	η	ν	c_4	ε	$\{2c_6\}$	κ	$\bar{\kappa}$	$\{8\Delta\}$	$\{\eta\Delta\}$	$\{2\nu\Delta\}$	q	$\{c_4\Delta\}$

36	48	51	56	60	72	80	84
$\{2c_6\Delta\}$	$\{4\Delta^2\}$	$\{\nu\Delta^2\}$	$\{c_4\Delta^2\}$	$\{2c_6\Delta^2\}$	$\{8\Delta^3\}$	$\{c_4\Delta^3\}$	$\{2c_6\Delta^3\}$
96	97	99	104	104	108	110	120
$\{2\Delta^4\}$	$\{\eta\Delta^4\}$	$\{\nu\Delta^4\}$	$\{\varepsilon\Delta^4\}$	$\{c_4\Delta^4\}$	$\{2c_6\Delta^4\}$	$\{\kappa\Delta^4\}$	$\{8\Delta^5\}$
123	128	128	132	144	147	152	156
$\{\nu\Delta^5\}$	$\{q\Delta^4\}$	$\{c_4\Delta^5\}$	$\{2c_6\Delta^5\}$	$\{4\Delta^6\}$	$\{\nu\Delta^6\}$	$\{c_4\Delta^6\}$	$\{2c_6\Delta^6\}$
168	176	180	192				
$\{8\Delta^7\}$	$\{c_4\Delta^7\}$	$\{2c_6\Delta^7\}$	$\{\Delta^8\}$				

Hereafter, we list the multiplications that are neither indicated in our chart, nor implied by the ring homomorphism $\pi_*(tmf) \to MF_*$. On the top are the generators, and on the bottom the degrees of the classes that support non-trivial multiplications by those generators (the images never involve $\eta^a c_4^b \Delta^c$ with $b \geq 1$, or $2c_4^a c_6 \Delta^b$):

ε	κ	$\bar{\kappa}$	q
1	1, 26.	1, 2, 8, 14, 15, 21, 22, 26, 98, 22, 32, 33, 34, 39, 46, 104, 110, 111, 116, 117, 118, 128, 129, 130, 135, 136, 142.	1, 3, 8, 14, 20, 21, 25, 27, 28, 34, 97, 99, 104, 110, 117, 118, 123, 124, 130.

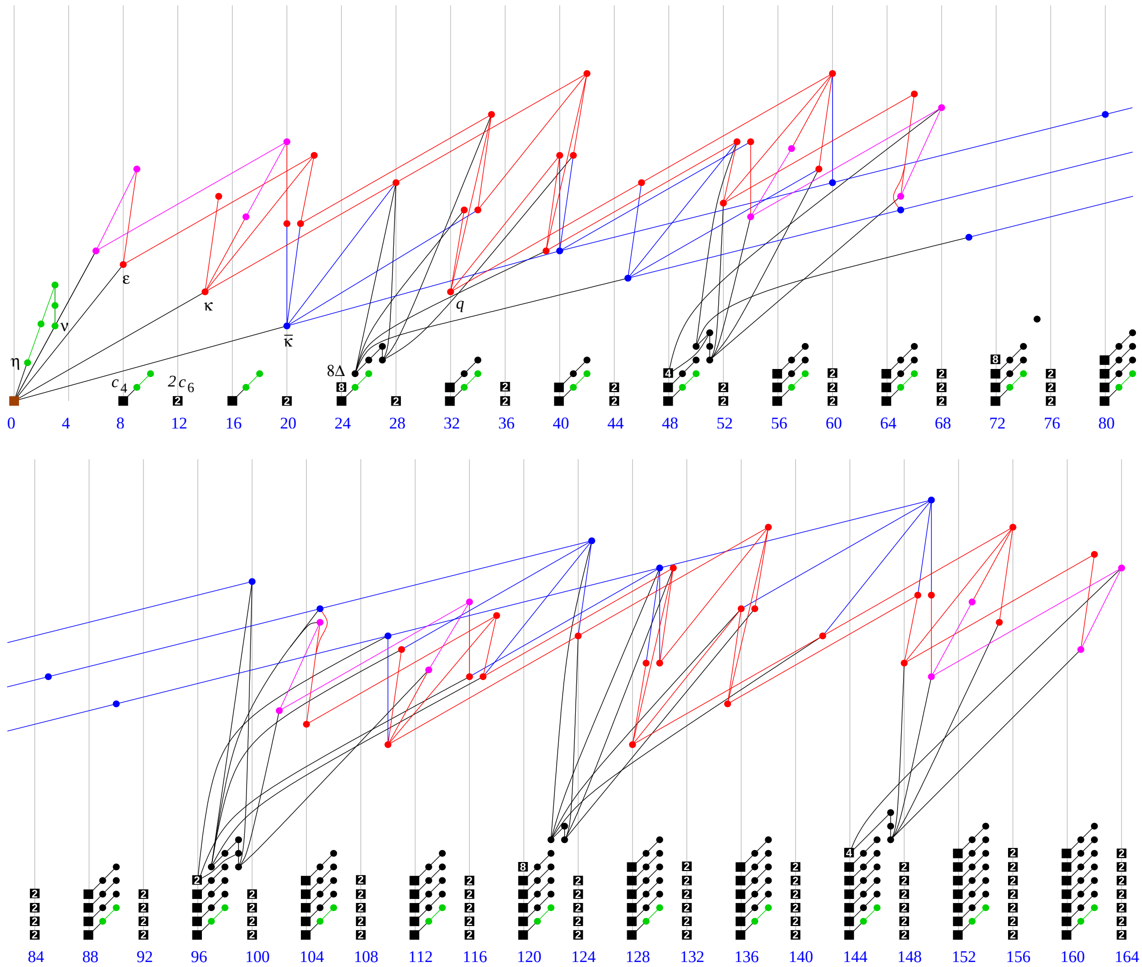
$\{\eta\Delta\}$	$\{2\nu\Delta\}$
1, 2, 3, 8, 14, 15, 20, 21, 25, 26, 27, 28, 32, 34, 35, 40, 41, 45, 50, 60, 65, 75, 80, 85, 97, 98, 99, 100, 104, 105, 110, 111, 113, 117, 122, 123, 124, 125, 128, 130, 131, 137.	1, 8, 14, 15, 25, 26, 27, 32, 33, 39, 96, 97, 98, 104, 110, 111, 122, 123, 128, 129, 135.

$\{\nu\Delta^2\}$	
1, 2, 3, 6, 8, 9, 14, 15, 17, 51, 54†, 65†, 96, 97, 98, 99, 102, 104, 105, 110, 111, 113, 116.	†: $\nu\{\nu\Delta^2\} \mapsto \nu^2\{\nu\Delta^4\}$. $\kappa\{\nu\Delta^2\} \mapsto \nu\kappa\{\nu\Delta^4\}$.

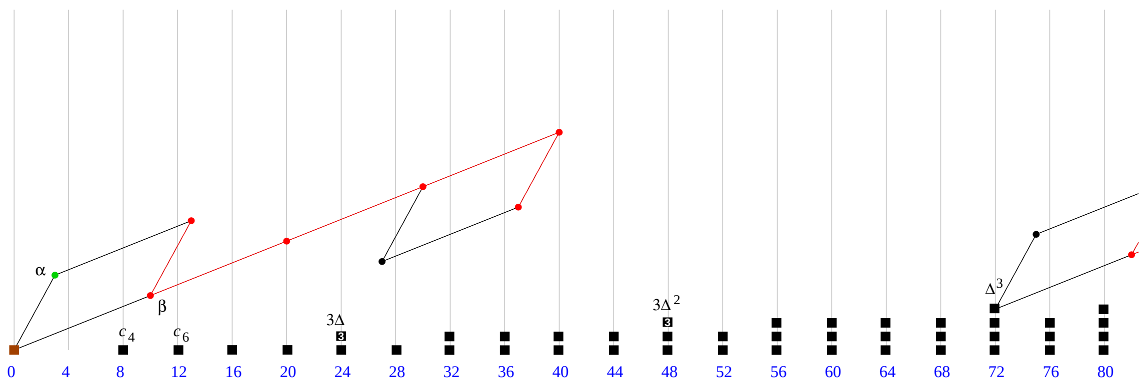
Acting with ε, κ, $\bar{\kappa}$, q, $\{\eta\Delta\}$, $\{2\nu\Delta\}$, $\{\nu\Delta^2\}$ on $\eta^a c_4^b \Delta^c$ with $b \geq 1$, or $2c_4^a c_6 \Delta^b$ always gives zero, except $\{\eta\Delta\}\eta^a c_4^b \Delta^c = \eta^{a+1} c_4^b \Delta^{c+1}$.

To finish, we emphasize the two relations that cannot be deduced by simply looking at our chart: $\{\eta\Delta\}^4 = \bar{\kappa}^5$, $\{2\nu\Delta\}^2 = \kappa\bar{\kappa}^2$.

The picture of $\pi_*(tmf)_{(2)}$ from the previous page being rather small, we include here a larger version:



The homotopy groups of *tmf* at the prime 3 exhibit similar phenomena as at the prime 2. The picture on the bottom of this page is an illustration of $\pi_*(tmf)_{(3)}$. The bullets represent $\mathbb{Z}/3\mathbb{Z}$'s and are named after the corresponding elements of $\pi_*(\mathbb{S})$. The squares represent $\mathbb{Z}_{(3)}$'s and are named after their image in MF_*. The slanted lines indicate multiplication by α and β. The top part of the



The homotopy groups of *tmf* at the prime 3

diagram is 72-periodic, with polynomial generator Δ^3. We have drawn the image of the Hurewicz in color: in green is the unique v_1-periodic class α, and in red are the v_2-periodic classes. The latter remain periodic in *tmf* through the identification $v_2^9 = \Delta^6$ (or maybe $v_2^{9/2} = \Delta^3$?). Once again, the white numbers in the squares indicate the size of $\operatorname{coker}(\pi_*(tmf) \to MF_*)_{(3)}$. The algebra $\pi_*(tmf)_{(3)}$ is finitely generated, with generators

degree:	3	8	10	12	24	27	32	36	48	56	60	72
name:	α	c_4	β	c_6	$\{3\Delta\}$	$\{\alpha\Delta\}$	$\{c_4\Delta\}$	$\{c_6\Delta\}$	$\{3\Delta^2\}$	$\{c_4\Delta^2\}$	$\{c_6\Delta^2\}$	$\{\Delta^3\}$

and many relations.

It is also worthwhile noting that the classes in dimensions 3, 13, 20, 30 (mod 72) support non-trivial $\langle \alpha, \alpha, - \rangle$ Massey products.

When localized at a prime $p \geq 5$, the homotopy ring of *tmf* becomes isomorphic to MF_*. Since $\Delta \in MF_{12}$ is a $\mathbb{Z}_{(p)}$-linear combination of c_4^3 and c_6^2, this ring then simplifies to $\pi_*(tmf)_{(p)} = \mathbb{Z}_{(p)}[c_4, c_6]$.

2. Localizations of *tmf*

The periodic version of *tmf* goes by the name *TMF*. Its homotopy groups are given by

$$\pi_*(TMF) = \pi_*(tmf)\big[\{\Delta^{24}\}^{-1}\big].$$

The groups $\pi_n(TMF)$ are finitely generated abelian groups, except for $n \equiv 0, 1, 2, 4 \pmod 8$ in which case they contain summands isomorphic to $\mathbb{Z}[x]$, $(\mathbb{Z}/2\mathbb{Z})[x]$, $(\mathbb{Z}/2\mathbb{Z})[x]$, and $\mathbb{Z}[x]$, respectively.

Fix a prime p, and let $K(n)$ denote the nth Morava K-theory at that prime (p is omitted from the notation). We can then consider the $K(n)$-localization $L_{K(n)}tmf$ of the spectrum *tmf*. The spectrum $L_{K(0)}tmf$ is simply the rationalisation of *tmf* (and does not depend on p). Its homotopy ring is therefore given by

$$\pi_*(L_{K(0)}tmf) = \pi_*(tmf) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The homotopy groups of $L_{K(1)}tmf$ are easiest to describe at the primes 2 and 3. In those cases, they are given by

$$\begin{aligned} \pi_*(L_{K(1)}tmf) &= \Big(\pi_*(KO)[j^{-1}]\Big)^{\hat{}}_p \\ &= \pi_*(KO)^{\hat{}}_p\,\langle j^{-1}\rangle \end{aligned} \qquad p = 2 \text{ or } 3. \tag{3}$$

Here, the notation $R\langle x\rangle$ refers to powers series $\sum_{k=0}^{\infty} a_k x^k$ whose coefficients $a_k \in R$ tend to zero p-adically as $k \to \infty$. The variable is called j^{-1} because its inverse corresponds to the j-invariant of elliptic curves. The reason why (3) is simpler at $p = 2$ and 3 is that, at those primes, there exists only one supersingular elliptic curve and its j-invariant is equal to zero. For general prime $p \geq 3$, let $\alpha_1, \ldots, \alpha_n$ denote the supersingular j-values. Each element α_i is a priori only an element of $\mathbb{F}_{p^n}$ (actually in $\mathbb{F}_{p^2}$), however, their union $S := \{\alpha_1, \ldots, \alpha_n\}$ is always a scheme over $\mathbb{F}_p$. Let $\tilde{S}$ denote any sheme over $\mathbb{Z}$ whose reduction mod p is S. The homotopy groups of $L_{K(1)}tmf$ are then given by

$$\pi_*(L_{K(1)}tmf) = \Big(\text{functions on } \mathbb{P}^1_{\mathbb{Z}} \setminus \tilde{S}\Big)^{\hat{}}_p\big[\,b^{\pm 1}\,\big], \qquad p \geq 3,$$

where b is a class in degree 4.

The homotopy ring of $L_{K(2)}tmf$ is the completion of $\pi_* TMF$ at the ideal generated by p and by the Hasse invariant E_{p-1}:

$$\pi_*(L_{K(2)}tmf) = \pi_*(TMF)^{\hat{}}_{(p,E_{p-1})}, \qquad p \text{ arbitrary},$$

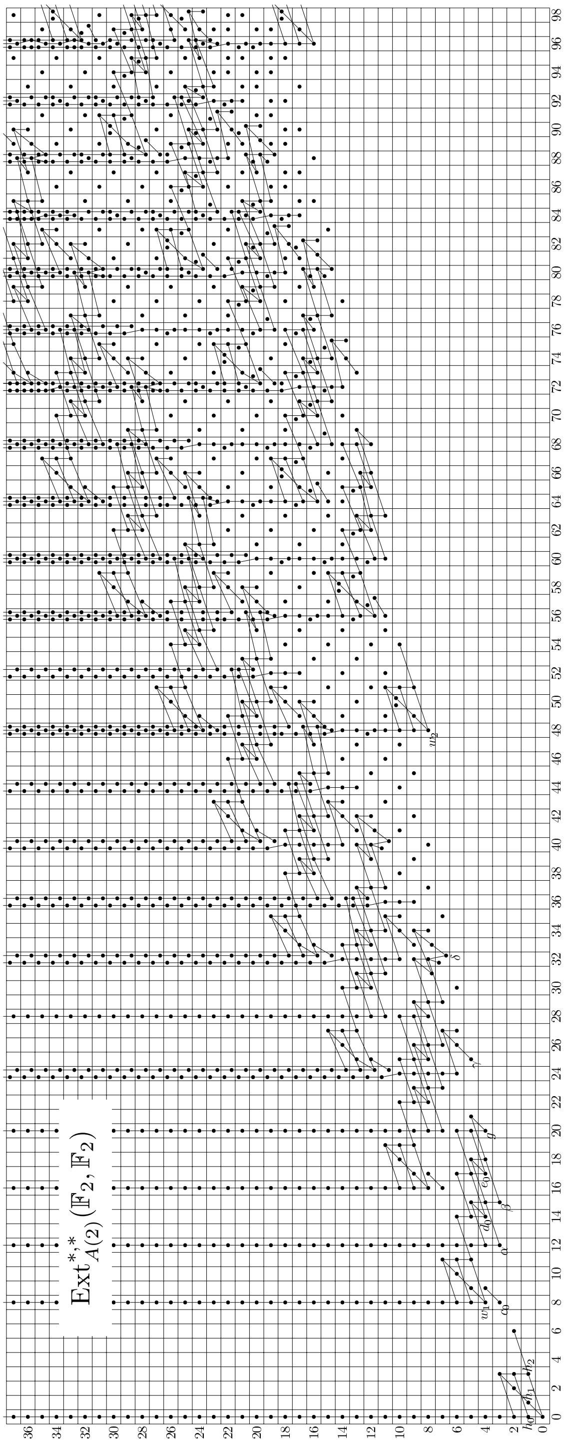
where the Hasse invariant is a polynomial in c_4 and c_6 whose zeroes correspond to the supersingular elliptic curves. Once again, given the fact that there is a unique supersingular elliptic curve at $p = 2$ and 3, the above formula simplifies to

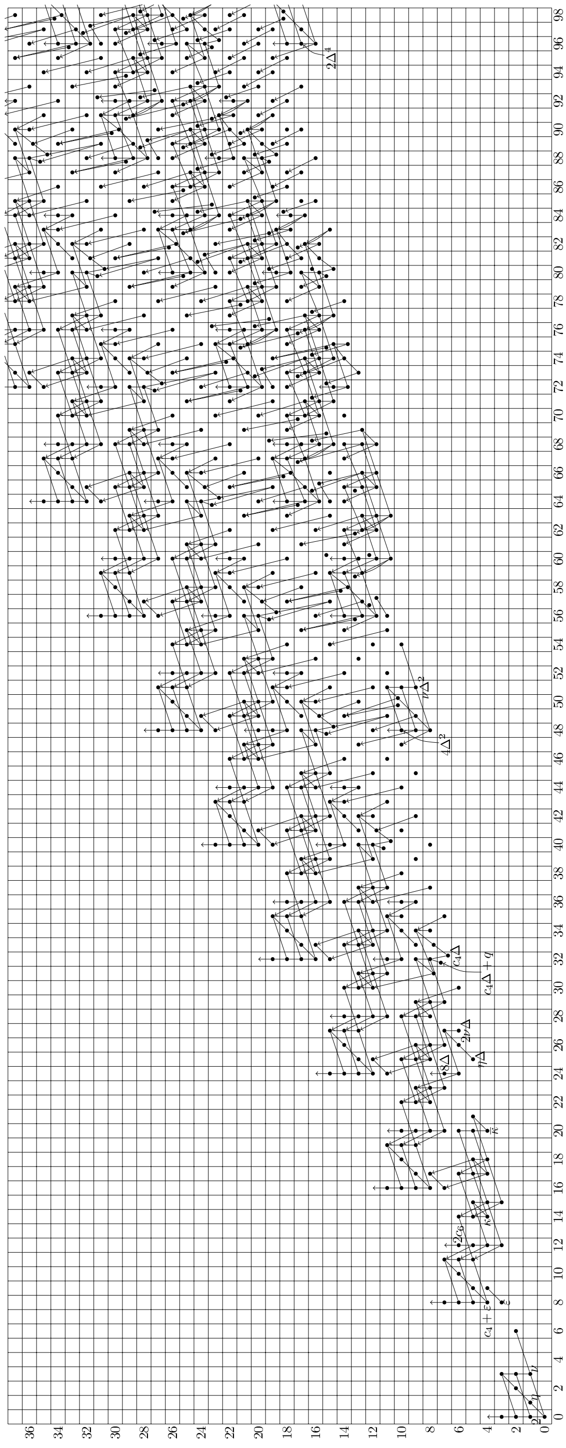
$$\pi_*(L_{K(2)}tmf) = \pi_*(TMF)^{\hat{}}_{(p,c_4)} \qquad p = 2, 3.$$

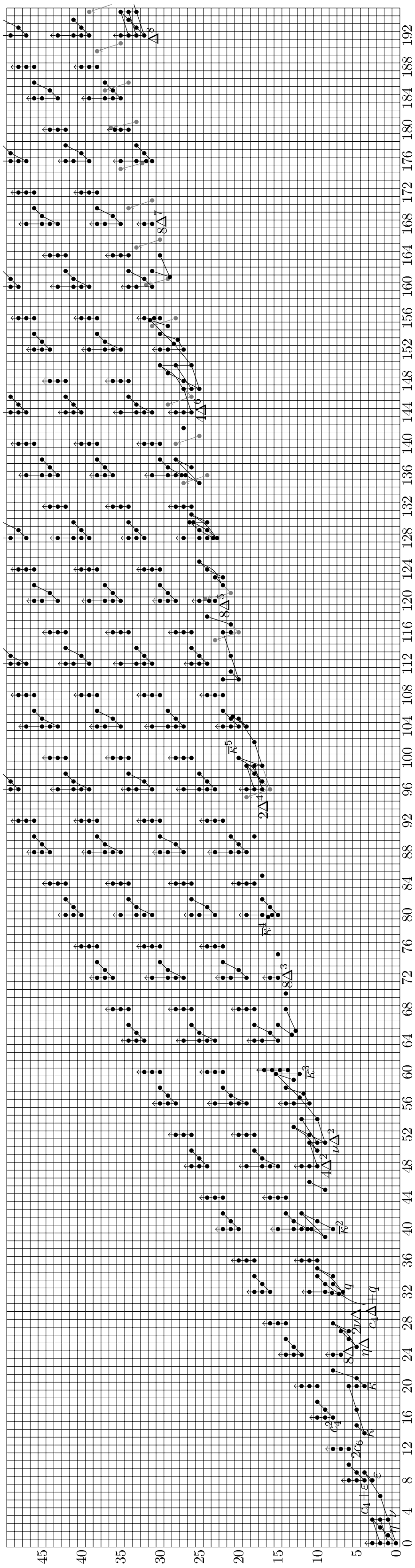
For $n > 2$, the localization $L_{K(n)}tmf$ is trivial, and therefore satisfies $\pi_*(L_{K(n)}tmf) = 0$.

3. The Adams spectral sequence

Given a fixed prime p, the Adams spectral sequence for tmf is a spectral sequence that converges to $\pi_*(tmf)^{\hat{}}_p$, see [1]. Its E_2 page is given by $\mathrm{Ext}_{A_p^{tmf}}(\mathbb{F}_p, \mathbb{F}_p)$, where A_p^{tmf} is a finite dimensional $\mathbb{F}_p$-







Above: The E_∞ page of the Adams spectral sequence for *tmf* at the prime 2.
Previous Page: The E_2 page with classes named as in (4); The differentials.

algebra that is a *tmf*-analog of the Steenrod algebra:

$$A_p^{tmf} := \hom_{tmf-\text{modules}}(H\mathbb{F}_p, H\mathbb{F}_p).$$

At the prime 2, the natural map $A_2^{tmf} \to A \equiv A_2$ to the Steenrod algebra is injective. Its image is the subalgebra $A(2) \subset A$ generated by Sq^1, Sq^2 and Sq^4. That algebra is of dimension 64 over $\mathbb{F}_2$, and defined by the relations

$$Sq^1Sq^1 = 0, \quad Sq^2Sq^2 = Sq^1Sq^2Sq^1,$$
$$Sq^1Sq^4 + Sq^4Sq^1 + Sq^2Sq^1Sq^2 = 0, \text{ and}$$
$$Sq^4Sq^4 + Sq^2Sq^2Sq^4 + Sq^2Sq^4Sq^2 = 0.$$

By the change of rings theorem, the Adams spectral sequence for *tmf* can then be identified with the classical Adams spectral sequence (see [5])

$$\begin{aligned} E_2 &= \mathrm{Ext}_A\big(H^*(tmf), \mathbb{F}_2\big) \\ &= \mathrm{Ext}_A\big(A/\!\!/A(2), \mathbb{F}_2\big) \Rightarrow \pi_*(tmf)\hat{{}_2}. \end{aligned}$$

The bigraded ring $\mathrm{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$ is generated by the classes:

(4)

bidegree:	(0,1)	(1,1)	(3,1)	(8,4)	(8,3)	(12,3)	(14,4)
name:	h_0	h_1	h_2	w_1	c_0	α	d_0
	(15,3)	(17,4)	(20,4)	(25,5)	(32,7)	(48,8)	
	β	e_0	g	γ	δ	w_2	

subject to the following complete set of relations:

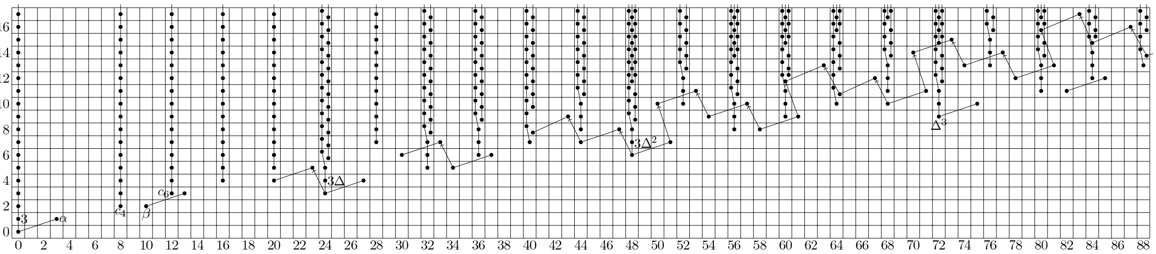
$h_0h_1 = 0$, $h_1h_2 = 0$, $h_0^2h_2 = h_1^3$, $h_0h_2^2 = 0$, $h_2^3 = 0$, $h_0c_0 = 0$, $h_1^2c_0 = 0$, $h_2c_0 = 0$, $c_0^2 = 0$, $c_0d_0 = 0$, $c_0e_0 = 0$, $c_0g = 0$, $c_0\alpha = h_0^2g$, $c_0\beta = 0$, $h_0^2d_0 = h_2^2w_1$, $h_1d_0 = h_0^2\beta$, $h_2d_0 = h_0e_0$, $d_0^2 = w_1g$, $d_0g = e_0^2$, $h_0\alpha d_0 = h_2\beta w_1$, $\alpha^2d_0 = \beta^2w_1$, $\beta d_0 = \alpha e_0$, $h_1e_0 = h_2^2\alpha$, $h_2e_0 = h_0g$, $\beta e_0 = \alpha g$, $h_1g = h_2^2\beta$, $h_2g = 0$, $e_0g = \alpha\gamma$, $g^2 = \beta\gamma$, $h_1\alpha = 0$, $h_1\beta = 0$, $h_2\alpha = h_0\beta$, $h_0\beta^2 = 0$, $h_2\beta^2 = 0$, $\alpha^4 = h_0^4w_2 + g^2w_1$, $h_0\gamma = 0$, $h_1^2\gamma = h_2\alpha^2$, $h_2\gamma = 0$, $c_0\gamma = h_1\delta$, $d_0\gamma = \alpha^2\beta$, $e_0\gamma = \alpha\beta^2$, $g\gamma = \beta^3$, $\gamma^2 = h_1^2w_2 + \beta^2g$, $h_0\delta = h_0\alpha g$, $h_1^2\delta = h_0d_0g$, $h_2\delta = 0$, $c_0\delta = 0$, $d_0\delta = 0$, $e_0\delta = 0$, $g\delta = 0$, $\alpha\delta = 0$, $\beta\delta = 0$ $\gamma\delta = h_1c_0w_2$, $\delta^2 = 0$.

The charts on the previous page only go until half the periodicity. Therefore, we have also included in the picture of the E_∞ page the most important differential in dimensions ≥ 96.

At the prime 3, the map from A_3^{tmf} to the Steenrod algebra not injective. Indeed, the algebra A_3^{tmf} is 24 dimensional, while its image in the Steenrod algebra is the 12 dmensional subalgebra generated by β and $\mathcal{P}^1$. Naming the generators by their images in A_3, the following relations define A_3^{tmf} (see [3]):

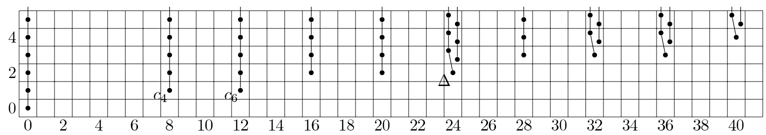
$$\beta^2 = 0, \quad (\mathcal{P}^1)^3 = 0,$$
$$\beta\,\mathcal{P}^1\beta\,\mathcal{P}^1 + \mathcal{P}^1\beta\,\mathcal{P}^1\beta = \beta\,(\mathcal{P}^1)^2\beta.$$

Note that the relation $\beta(\mathcal{P}^1)^2 + \mathcal{P}^1\beta\mathcal{P}^1 + (\mathcal{P}^1)^2\beta = 0$ holds in A_3, but not in A_3^{tmf}.



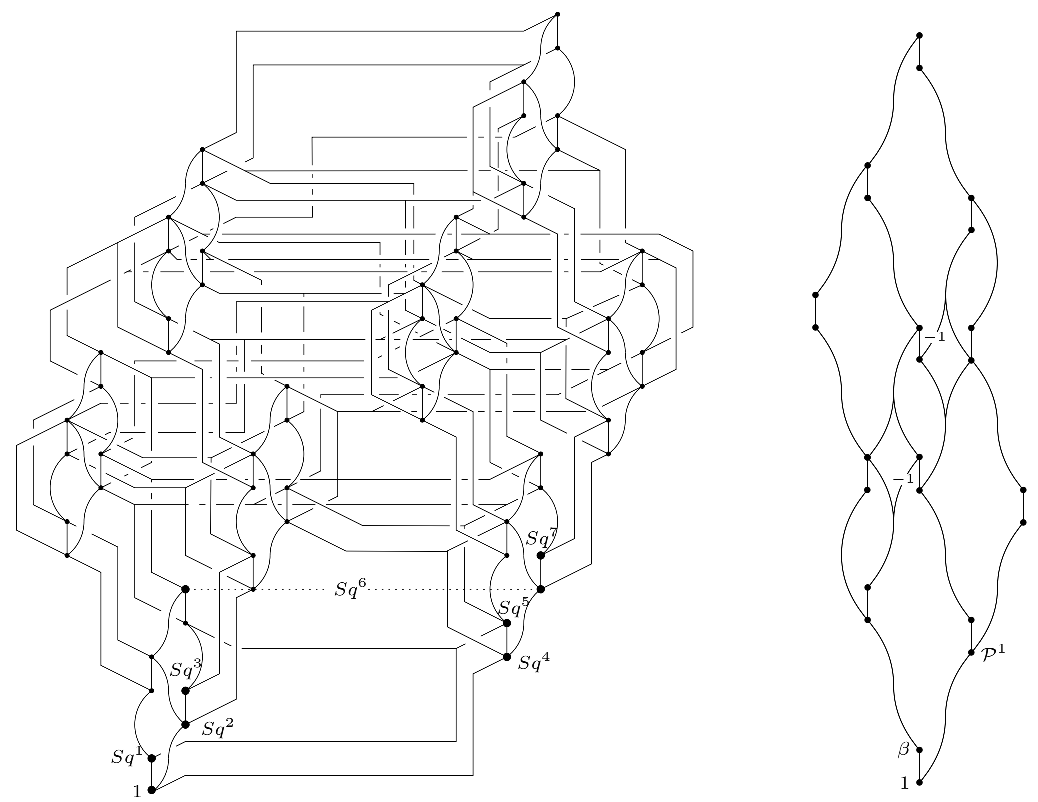
The Adams spectral sequence $\mathrm{Ext}_{A_3^{tmf}}(\mathbb{F}_3, \mathbb{F}_3) \Rightarrow \pi_*(tmf)\hat{}_3$.

At primes $p > 3$, the algebra A_p^{tmf} is an exterior algebra on generators in degrees 1, 9, and 13. The ring $\mathrm{Ext}_{A_p^{tmf}}(\mathbb{F}_p, \mathbb{F}_p)$ is a polynomial algebra on classes in bidegrees $(0,1)$, $(8,1)$, and $(12,1)$ and the Adams spectal sequence for *tmf* collapses.



The Adams spectral sequence for *tmf* at a prime $p > 3$.

It is interesting to note that regardless of the prime, the algebra A_p^{tmf} has its top dimensional class in degree 23. Below, we picture the algebras A_p^{tmf} for the primes 2 and 3:



4. The elliptic spectral sequence

The spectrum *TMF* is the global sections of a sheaf of E_∞ ring spectra over the moduli space of elliptic curves $\mathcal{M}_{ell}$. As far as we know, there is no moduli space yielding *tmf* that way. To construct *tmf*, one first considers the sheaf $\mathcal{O}^{\mathrm{top}}$ of E_∞ ring spectra over the Deligne-Mumford compactification $\overline{\mathcal{M}}_{ell}$. The spectrum of global sections

$$Tmf := \Gamma(\overline{\mathcal{M}}_{ell}; \mathcal{O}^{\mathrm{top}})$$

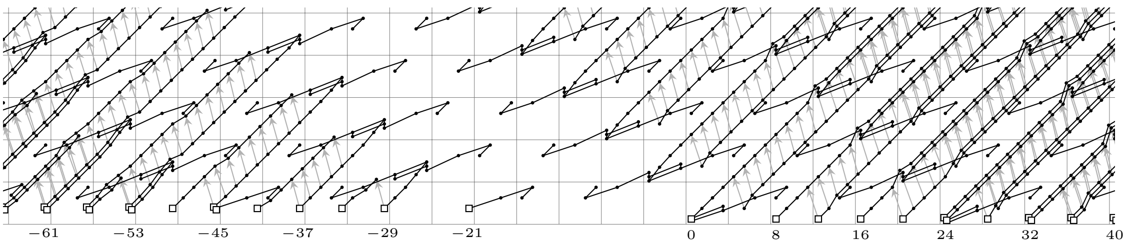
is not connective, and its connective cover is *tmf*. One can also recover $\pi_*(Tmf)$ from $\pi_*(tmf)$ by the following "Serre duality" short exact sequence:

$$0 \to \mathrm{Ext}^1\big(\pi_{n-22}(tmf), \mathbb{Z}\big) \to \pi_{-n}(Tmf) \to \mathrm{Hom}\big(\pi_{n-21}(tmf), \mathbb{Z}\big) \to 0, \qquad n > 0.$$

The "elliptic spectral sequence" is the descent spectral sequence

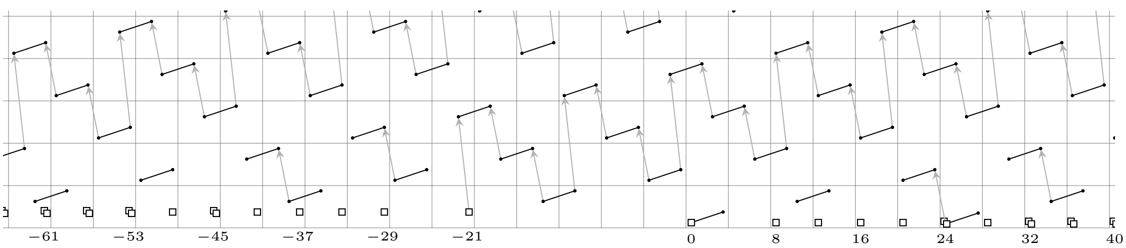
$$H^s\big(\overline{\mathcal{M}}_{ell}; \pi_t\mathcal{O}^{\mathrm{top}}\big) = H^s\left(\overline{\mathcal{M}}_{ell}\,; \begin{Bmatrix} \omega^{t/2} & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{Bmatrix}\right) \Rightarrow \pi_{t-s}(Tmf).$$

Its E_2 page at $p = 2$ is as follows (see [2, 4]):



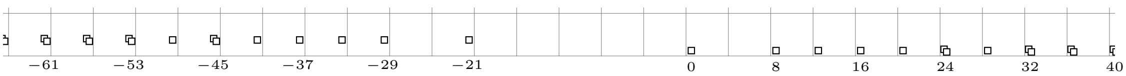
In the above chart, squares indicate copies of $\mathbb{Z}_{(2)}$ and bullets indicate copies of $\mathbb{Z}/2\mathbb{Z}$. Two (three) bullets stacked vertically onto each other indicate a copy of $\mathbb{Z}/4\mathbb{Z}$ ($\mathbb{Z}/8\mathbb{Z}$). The d_3 differentials are drawn in gray; the remaining differentials $d_5, d_7, d_9, \ldots, d_{25}$ are drawn on the charts on pages 9 – 12. The colors on those charts indicate the periodicity by which the various sub-patterns of differentials repeat.

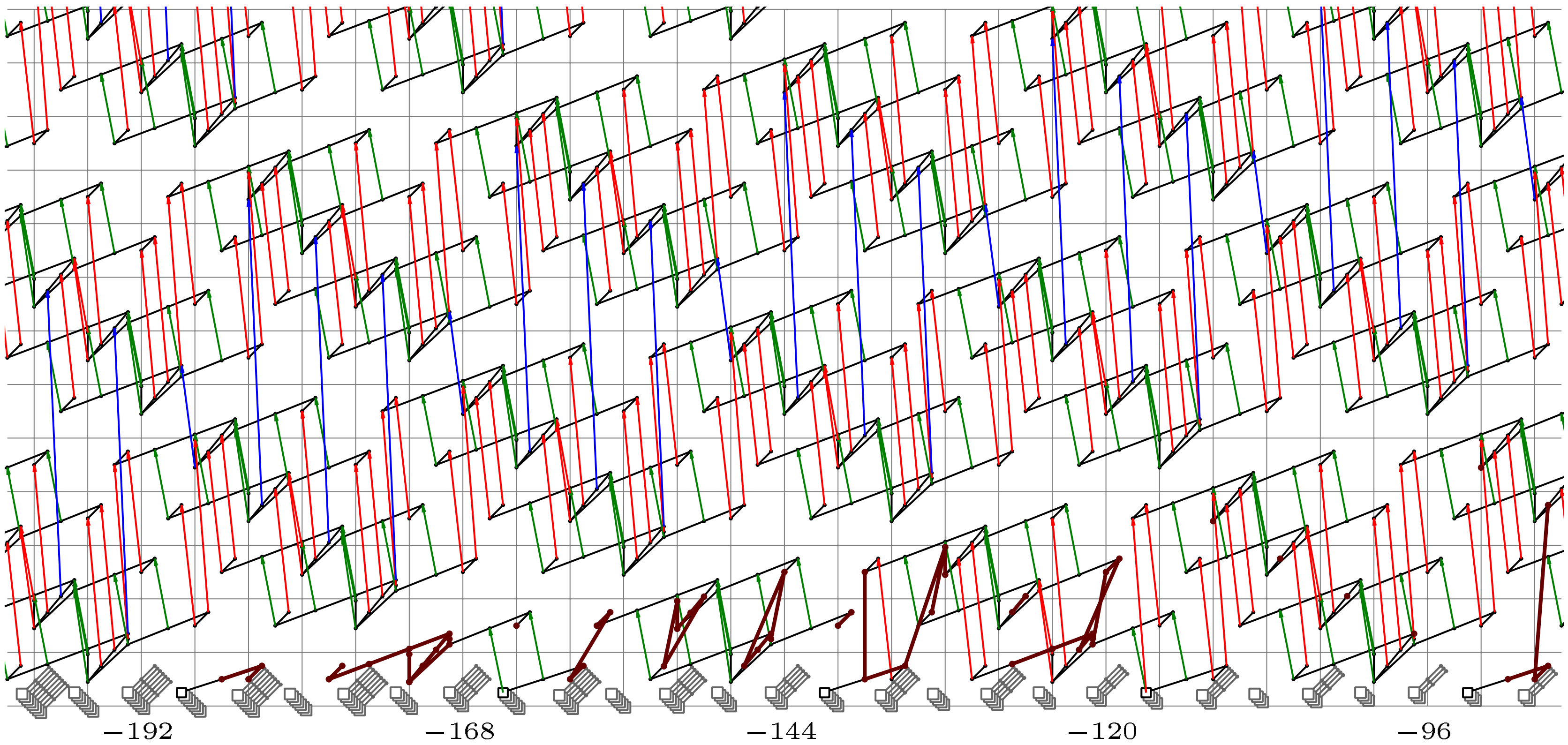
At $p = 3$, the elliptic spectral sequence looks as follows:



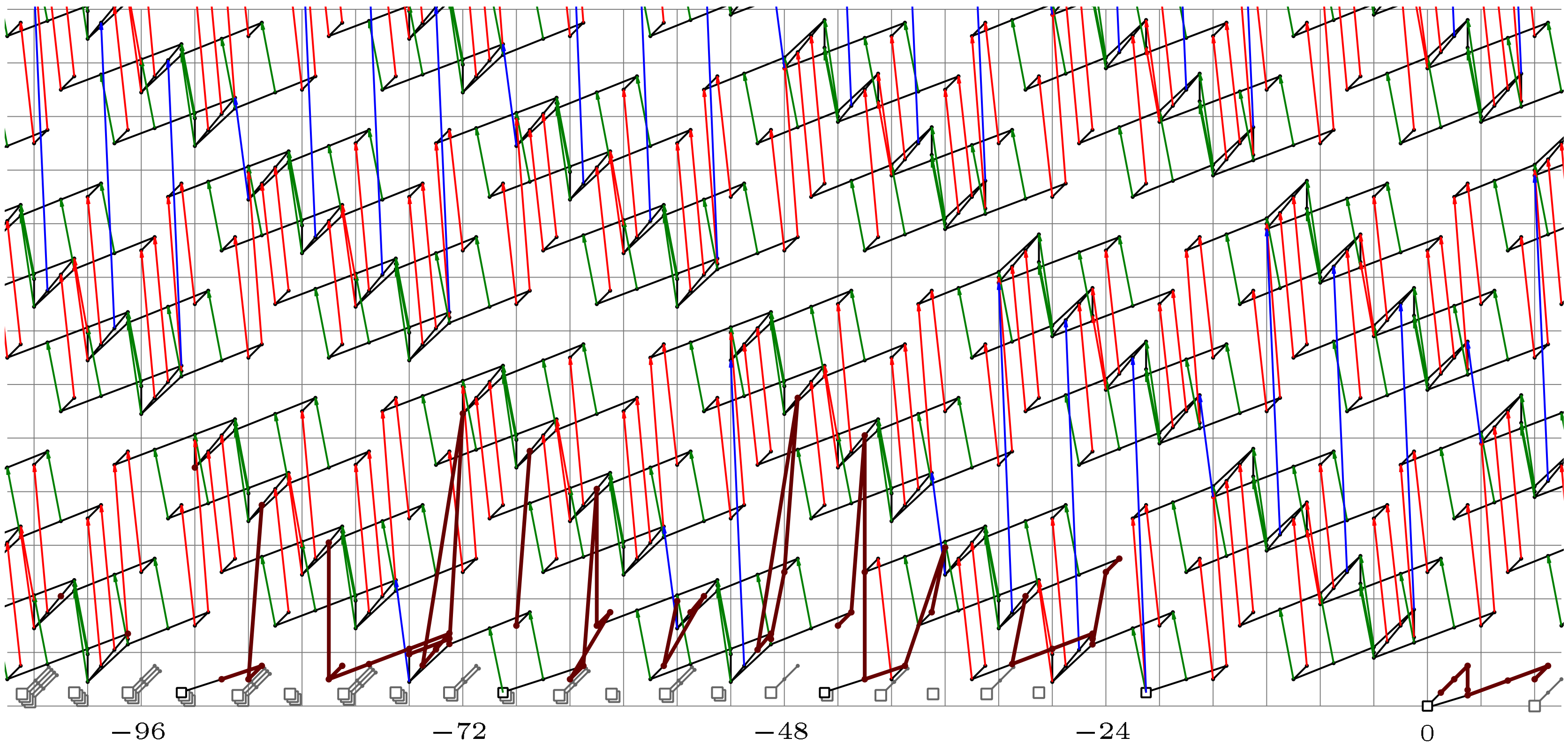
The squares indicate copies of $\mathbb{Z}_{(3)}$, and the bullets indicate copies of $\mathbb{Z}/3\mathbb{Z}$.

At $p \geq 5$, there is no p-torsion and the elliptic spectral sequence collapses:

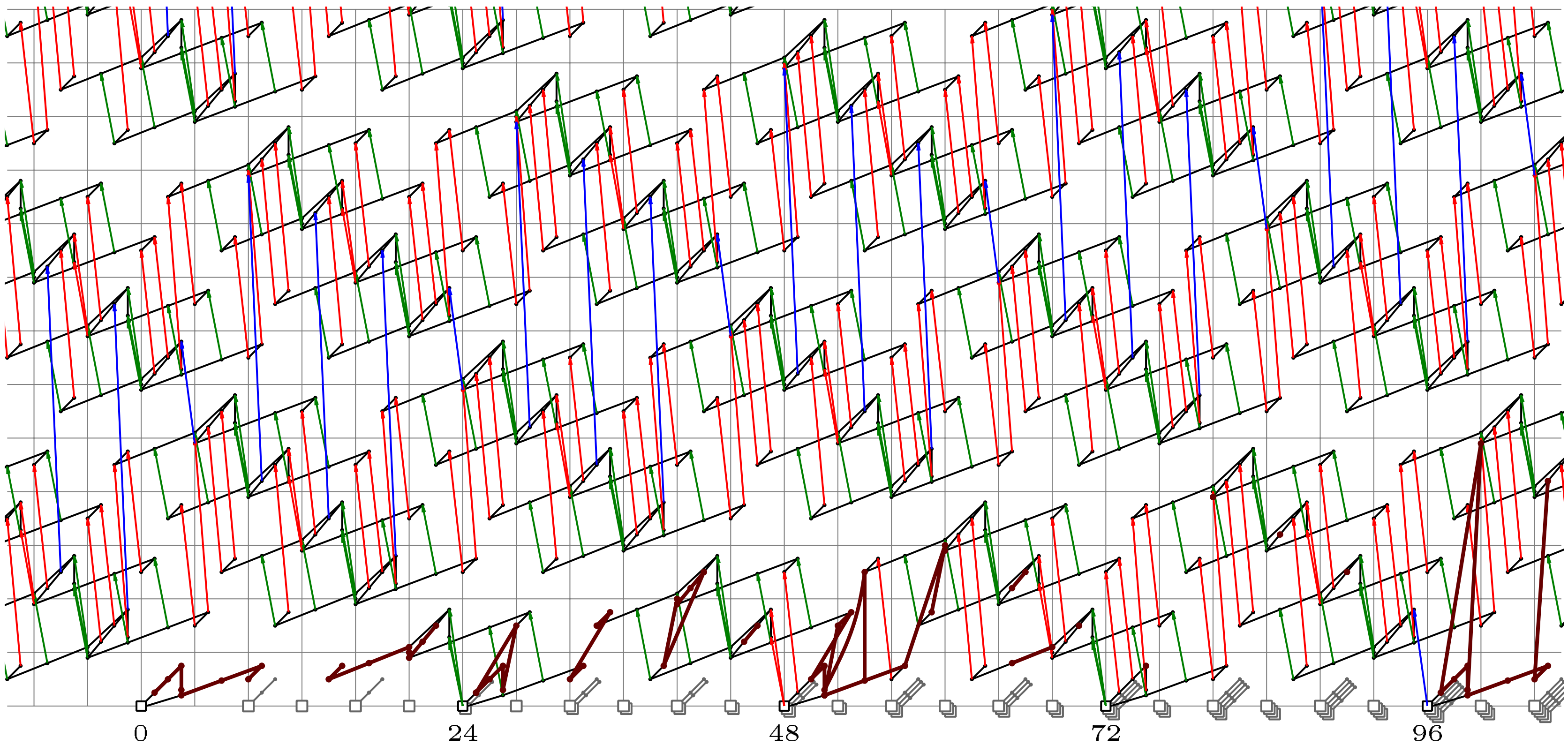




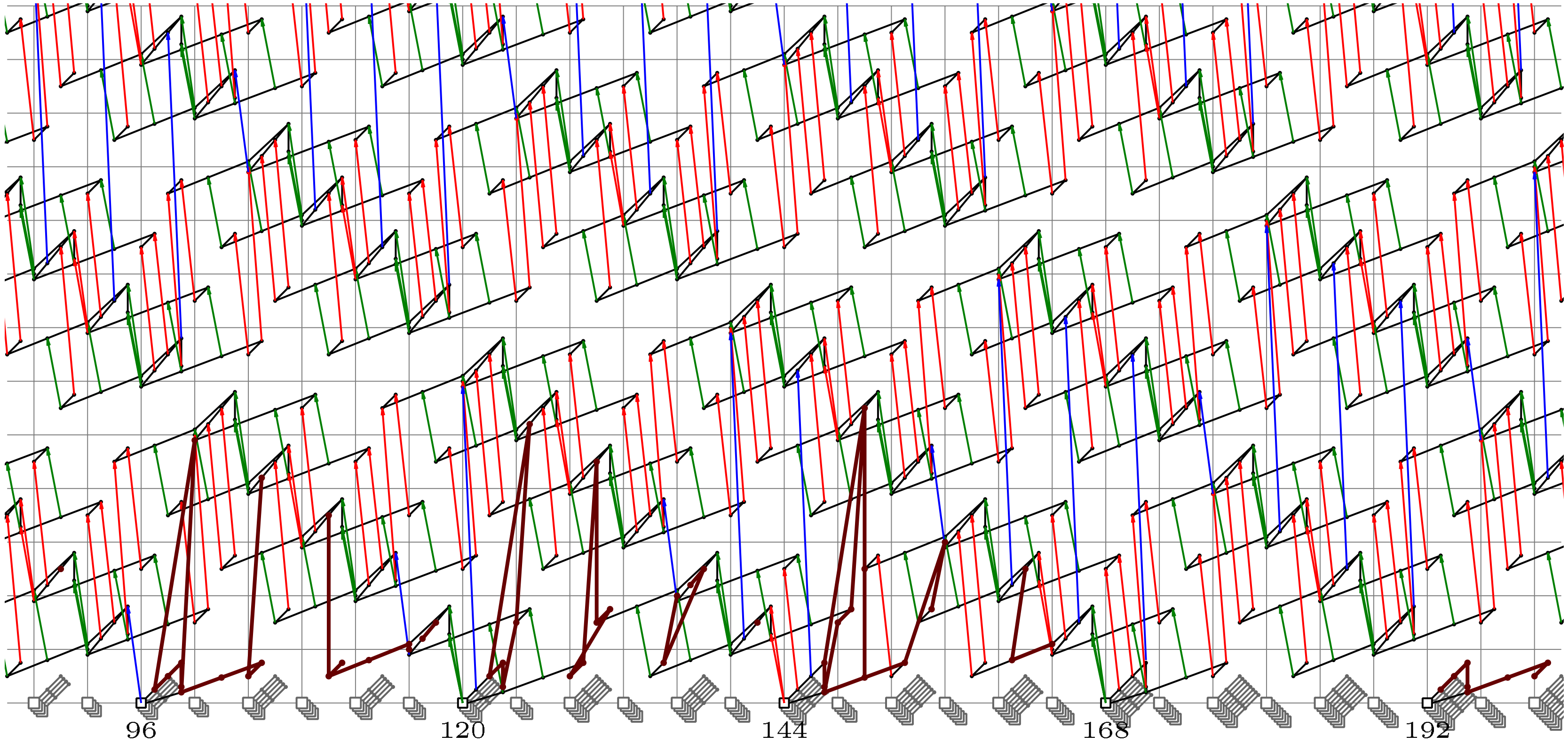
The elliptic spectral sequence $H^q(\overline{\mathcal{M}}_{Ell};\omega^{\otimes p}) \Rightarrow \pi_{2p-q}(Tmf)$ for $2p-q$ between -202 and -86.



The elliptic spectral sequence $H^q(\overline{\mathcal{M}}_{Ell};\omega^{\otimes p}) \Rightarrow \pi_{2p-q}(Tmf)$ for $2p-q$ between -106 and 10.



The elliptic spectral sequence $H^q(\overline{\mathcal{M}}_{Ell}; \omega^{\otimes p}) \Rightarrow \pi_{2p-q}(Tmf)$ for $2p-q$ between -10 and 106.



The elliptic spectral sequence $H^q(\overline{\mathcal{M}}_{Ell};\omega^{\otimes p}) \Rightarrow \pi_{2p-q}(Tmf)$ for $2p-q$ between 86 and 202.

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Conjecture (Mark Mahowald)**.** *The image of the Hurewicz homomorphism $\pi_*(\mathbb{S}) \to \pi_*(tmf)$ is given by the classes drawn in color in the pictures on page 3.*

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