

BOUSFIELD LOCALIZATION AND THE HASSE SQUARE

TILMAN BAUER

1. BOUSFIELD LOCALIZATION

The general idea of localization at a spectrum E is to associate to any spectrum X the “part of X that E can see”, denoted by $L_E X$. In particular, it is desirable that L_E is a functor with the following equivalent properties:

$$E \wedge X \simeq * \implies L_E X \simeq *$$

If $X \rightarrow Y$ induces an equivalence $E \wedge X \rightarrow E \wedge Y$ then $L_E X \xrightarrow{\sim} L_E Y$.

Definition 1.1. A spectrum X is called *E-acyclic* if $E \wedge X \simeq *$. It is called *E-local* if for each *E-acyclic* T , $[T, E] = 0$, where $[T, E]$ denotes the group of stable homotopy classes. A map of spectra $f: X \rightarrow Y$ is called an *E-equivalence* if $E \wedge f: E \wedge X \rightarrow E \wedge Y$ is a homotopy equivalence. It is immediate that a spectrum X is *E-local* iff for each *E-equivalence* $S \rightarrow T$, the induced map $[T, X] \rightarrow [S, X]$ is an isomorphism.

A spectrum Y with a map $X \rightarrow Y$ is called an *E-localization* of X if Y is *E-local* and $X \rightarrow Y$ is an *E-equivalence*.

If a localization of X exists, then it is unique up to homotopy and will be denoted by $X \xrightarrow{\eta_E} L_E X$.

Localizations of this kind were first studied by Adams [Ada74], but set-theoretic difficulties prevented him from actually constructing them. Bousfield found a way of overcoming these problems in the unstable category [Bou75]; for spectra, he showed in [Bou79] that localization functors exist for arbitrary E .

We start by collecting a couple of easy facts about localizations.

Lemma 1.2. *Module spectra over a ring spectrum E are E -local.*

Proof. Since any map from a spectrum W into an E -module spectrum M can be factored through $E \wedge W$, it follows that all maps from an *E-acyclic* W into M are nullhomotopic. \square

Lemma 1.3. *If $v \in \pi_*(E)$ is an element of a ring spectrum E (of an arbitrary but homogeneous degree), then $L_E \simeq L_{v^{-1}E \vee E/v}$, where E/v denotes the cofiber of multiplication with v and*

$$v^{-1}E = \operatorname{colim} \left(E \xrightarrow{v} E \xrightarrow{v} \cdots \right)$$

the mapping telescope.

Proof. It suffices to show that the class of *E-acyclics* agrees with the class of $(v^{-1}E \vee E/v)$ -acyclics. Since $L_{v^{-1}E \vee E/v}$ is a module spectrum over E , *E-acyclics* are clearly $(v^{-1}E \vee E/v)$ -acyclic; conversely, if $E/v \wedge W \simeq *$ then $v: E \wedge W \rightarrow E \wedge W$ is a homotopy equivalence, hence $E \wedge W \simeq v^{-1}E \wedge W$. Thus if also $v^{-1}E \wedge W \simeq *$, W is *E-acyclic*. \square

Lemma 1.4. *Homotopy limits and retracts of E -local spectra are E -local.*

Proof. The statement about retracts is obvious. For the statement about limits, first observe that a spectrum X is *E-local* if and only if the mapping spectrum $\operatorname{Map}(T, X)$ is contractible for all *E-acyclic* T . This is obvious because $\pi_k \operatorname{Map}(T, X) = [\Sigma^k T, X]$, and if T is *E-acyclic* then so are all its suspensions.

Now if $F: I \rightarrow \{\text{spectra}\}$ is a diagram of E -local spectra, the claim follows from the equivalence

$$\text{Map}(T, \text{holim } F) \simeq \text{holim } \text{Map}(T, F)$$

□

The following lemma characterizes E -localizations.

Lemma 1.5. *The following are equivalent for a map of spectra $X \rightarrow Y$:*

- $X \rightarrow Y$ is an E -localization;
- ① $X \rightarrow Y$ is the initial map into an E -local target;
- ② $X \rightarrow Y$ is the terminal map which is an E -equivalence.

Proof. Obvious from the axioms. □

This characterization suggests two ways of constructing $X \rightarrow L_E X$:

- ① $L_E X = \underset{\substack{X \rightarrow Y \\ Y \text{ } E\text{-local}}}{\text{holim}} Y$ or
- ② $L_E X = \underset{\substack{X \rightarrow Y \\ E\text{-equivalence}}}{\text{hocolim}} Y$.

In both cases, these limits are not guaranteed to exist because the indexing categories are not small. This is more than a set-theoretic nuisance and requires a deeper study of the structure of the background categories.

I will first briefly discuss what can be done with approach ①. The main construction will be closer to method ②.

① **Localizations as limits.** For a ring spectrum E , instead of indexing the homotopy limit over all $X \rightarrow Y$ with Y E -local, we could use the spaces in the Adams tower for E :

$$X \rightarrow \text{Tot}^n \left(E^{\wedge(\bullet+1)} \wedge X \right),$$

which is a subdiagram because $E \wedge X$ is E -local for any X by Lemma 1.2, and E -locality satisfies the 2-out-of-3 property for cofibration sequences of spectra. For this cosimplicial construction to make sense, the ring spectrum E has to be associative in a strict sense (e.g. in the framework of [EKMM97]) or at least A_∞ [BL10]), or one can restrict to *cofacial* spectra: A cofacial spectrum is a functor from Δ_f to spectra, where Δ_f is the subcategory of Δ with the same objects but only injective maps. In that case, Tot is just defined as the homotopy limit, and one can show that this agrees with the cosimplicial Tot if the cofacial spectrum is the underlying cofacial spectrum of a cosimplicial spectrum. Note that in this approach, no multiplication on E is needed whatsoever – this works with any coaugmented spectrum $\mathbb{S} \rightarrow E$.

If we are lucky, $X \rightarrow X_{\hat{E}} =_{\text{def}} \text{Tot}(E^{\wedge(\bullet+1)} \wedge X)$ is an E -localization. This is not always the case – $X \rightarrow X_{\hat{E}}$ sometimes fails to be an E -equivalence. Whether or not $L_E X \simeq X_{\hat{E}}$, the latter is what the E -based Adams-Novikov spectral sequence converges to and thus is of independent interest. If $L_E X$ can be built from E -module spectra by a finite sequence of cofiber extensions and retracts, then $L_E X \simeq X_{\hat{E}}$ [Bou79, Thm 6.10] (such spectra are called E -prenilpotent). For some spectra E , every X is E -prenilpotent; these spectra have the characterizing property that their Adams spectral sequence has a common horizontal vanishing line at E_∞ and a horizontal stabilization line at every E_r for every finite CW-spectrum [Bou79, Thm 6.12]. A necessary condition for this is that E is *smashing*, i.e., that $L_E X = X \wedge L_E S^0$ for every spectrum X .

② **Localizations as colimits.** Bousfield's approach to constructing localizations uses colimits. The basic idea for cutting down the size of the diagram the colimit is formed over is the following observation:

To check if X is E -local, it is enough to show that for any E -equivalence $S \rightarrow T$ with $\#S, \#T < \kappa$ for some cardinal κ depending only on E , $[T, X] \xrightarrow{\cong} [S, X]$.

At this point, it is not crucial what exactly we mean by $\#S$. For a construction of $L_E X$ that is functorial up to homotopy, it is enough to define $\#S$ to be the number of stable cells.

Given this observation, $L_E X$ can be constructed in a small-object-argument-like fashion by forming homotopy pushouts

$$\begin{array}{ccc} \coprod_{\substack{S \rightarrow T \\ E\text{-eq.}}} S & \longrightarrow & X \\ \downarrow & & \downarrow \\ \coprod_{\substack{S \rightarrow T \\ E\text{-eq.}}} T & \longrightarrow & X_{(1)} \end{array}$$

and iterating this transfinitely (using colimits at limit ordinals). When the cardinal κ is reached, $X_{(\kappa)}$ is E -local because it satisfies the lifting condition for “small” $S \rightarrow T$.

Theorem 1.6. *The category of spectra has a model structure with*

- *cofibrations the usual cofibrations of spectra, i.e. levelwise cofibrations $A_n \rightarrow B_n$ such that*

$$\mathbb{S}^1 \wedge B_n \cup_{\mathbb{S}^1 \wedge A_n} A_{n+1} \rightarrow B_{n+1}$$

- are also cofibrations;*
- *weak equivalences the (stable) E -equivalences;*
- *fibrations given by the lifting property*

The fibrant objects in this model structure are the E -local Ω -spectra.

Here are some explicit examples of localization functors.

- Example 1.7.* (1) $E = \mathbb{S}^0$. In this case, L_E is the functor that replaces a spectrum by an equivalent Ω -spectrum.
- (2) $E = M(\mathbb{Z}_{(p)}) = \text{Moore spectrum}$. In this case $L_E X \simeq X_{(p)}$ is the Bousfield p -localization. This is an example of a *smashing localization*, i.e. $L_E X \simeq X \wedge L_E \mathbb{S}^0$, which in this case is also the same as $X \wedge E$.
- (3) $E = M(\mathbb{Z}/p)$. For connective X , $L_E X \simeq X_p^\wedge$ is the p -completion functor

$$X_p^\wedge = \text{holim}\{\cdots \rightarrow X \wedge M(\mathbb{Z}/p^2) \rightarrow X \wedge M(\mathbb{Z}/p)\}.$$

We write $X \xrightarrow{\eta_p} L_p X$ for this localization $X \xrightarrow{\eta_E} L_E X$.

- (4) $E = M(\mathbb{Q}) = H\mathbb{Q}$. As in (2), $L_E X = X \wedge L_{\mathbb{Q}} \mathbb{S}^0 = X \wedge H\mathbb{Q}$ is smashing; it is the rationalization of X . As in the previous case, we write $X \xrightarrow{\eta_{\mathbb{Q}}} L_{\mathbb{Q}} X$ for this localization $X \xrightarrow{\eta_E} L_E X$.

2. THE SULLIVAN ARITHMETIC SQUARE

The arithmetic square is a homotopy cartesian square that allows one to reconstruct a space if, roughly, all of its mod- p -localizations and its rationalization are known. For the case of nilpotent spaces, which is similar to spectra, this was first observed by Sullivan [Sul05].

Lemma 2.1. *For any spectrum X , the following diagram is a homotopy pullback square:*

$$\begin{array}{ccc} X & \xrightarrow{\Pi \eta_p} & \prod_p L_p X \\ \eta_{\mathbb{Q}} \downarrow & & \downarrow \eta_{\mathbb{Q}} \\ L_{\mathbb{Q}} X & \xrightarrow{L_{\mathbb{Q}}(\Pi \eta_p)} & L_{\mathbb{Q}} \left(\prod_p L_p X \right) \end{array}$$

This is a special case of

Proposition 2.2. *Let E, F, X be spectra with $E_*(L_F X) = 0$. Then there is a homotopy pullback square*

$$\begin{array}{ccc} L_{E \vee F} X & \xrightarrow{\eta_E} & L_E X \\ \eta_F \downarrow & & \downarrow \eta_F \\ L_F X & \xrightarrow{L_F(\eta_E)} & L_F L_E X \end{array}$$

In the case of Prop. 2.1, $E = \bigvee_p M(\mathbb{Z}/p)$, $F = H\mathbb{Q}$. To see that $L_E = \prod_p L_p$, we have to show that there are no nontrivial homotopy classes from an E -acyclic space to a spectrum of the form $\prod_p L_p X$, which is immediate, and that

$$M(\mathbb{Z}/p)_*(X) \xrightarrow{\cong} M(\mathbb{Z}/p)_* \left(\prod_l L_l X \right)$$

is an isomorphism for all p . The latter holds because smashing with $M(\mathbb{Z}/p)$ commutes with products since $M(\mathbb{Z}/p)$ is a finite (two-cell) spectrum (use Spanier-Whitehead duality).

Furthermore, the condition $E_*(L_F X) = E_*(H\mathbb{Q} \wedge X) = 0$ is satisfied because $H_*(M(\mathbb{Z}/p); \mathbb{Q}) = 0$.

Proof of the proposition. Note that the map denoted η_E in the diagram is the unique factorization of $\eta_E: X \rightarrow L_E X$ through $L_{E \vee F} X$, which exists because $X \rightarrow L_{E \vee F} X$ is an E -equivalence. The same holds for η_F , and furthermore, these maps are E - and F -equivalences, respectively. Now let P be the pullback. We need to see that (1) P is $(E \vee F)$ -local and (2) the induced map $X \rightarrow P$ is an E - and an F -equivalence. For (1), take a spectrum T with $E_* T = F_* T = 0$. Then in the Mayer-Vietoris sequence for the pullback,

$$\cdots \rightarrow [T, P] \rightarrow [T, L_E X] \oplus [T, L_F X] \rightarrow [T, L_F L_E X] \rightarrow \cdots,$$

the two terms on the right are zero, hence so is $[T, P]$.

For (2), observe that $P \rightarrow L_F X$ is an F -equivalence because it is the pullback of η_F on $L_E X$, and since $X \rightarrow L_F X$ is also an F -equivalence, so is $X \rightarrow P$. The same argument works for $P \rightarrow L_E X$ except that here, the bottom map is an E -equivalence for the trivial reason that both terms are E -acyclic by the assumption. \square

3. MORAVA K -THEORIES AND RELATED RING SPECTRA

Given a complex oriented even ring spectrum E and an element $v \in \pi_* E$, we would like to construct a new complex oriented ring spectrum E/v such that $\pi_*(E/v) = (\pi_* E)/(v)$. This is clearly not always possible. The machinery of commutative \mathbb{S} -algebras of [EKMM97] (or any other construction of a symmetric monoidal category of spectra, such as symmetric spectra) allows us to make this work in many cases where more classical homotopy theory has to rely on ad-hoc constructions (such as the Baas-Sullivan theory of bordism of manifolds with singularities).

In this section, let E be a complex oriented even commutative \mathbb{S} -algebra and A an E -module spectrum with a commutative ring structure in the homotopy category of E -modules, and which is also a complex oriented even ring spectrum. Let us call this an E -even ring spectrum. A commutative E -algebra would of course be fine, but we need the greater generality.

Theorem 3.1 ([EKMM97, Chapter V]). *For any $v \in \pi_* E$, $v^{-1}A$ carries the structure of an E -even ring spectrum. Furthermore, if v is a non-zero divisor then A/v is also an E -even ring spectrum.*

Even if A is a commutative E -algebra (for example, $A = E$), the resulting spectrum is usually not a commutative \mathbb{S} -algebra.

Of course, this construction can be iterated to give

Corollary 3.2. *Given a graded ideal $I \triangleleft \pi_* E$ generated by a regular sequence and a graded multiplicative set $S \subset \pi_* E$, one can construct an E -even ring spectrum $S^{-1}A/I$ with $\pi_* S^{-1}A/I = S^{-1}(\pi_* A)/I$.*

In particular, this can be done for $E = MU$. For example, BP can be constructed in this way by taking $I = \ker(MU_* \rightarrow BP_*)$, which is generated by a regular sequence. It is currently not known whether BP is a commutative \mathbb{S} -algebra. However, the methods above allow us to construct all the customary BP -ring spectra by pulling regular sequence back to $E = MU_*$ and letting $A = BP$. For example,

$$\begin{aligned} E(n) &= v_n^{-1}BP/(v_{n+1}, v_{n+2}, \dots) \\ K(n) &= v_n^{-1}BP/(p, v_1, \dots, v_{n-1}, v_{n+1}, \dots) \\ P(n) &= BP/(p, v_1, \dots, v_{n-1}) \\ B(n) &= v_n^{-1}BP/(p, v_1, \dots, v_{n-1}). \end{aligned}$$

Any MU -even ring spectrum A gives rise to a Hopf algebroid (A_*, A_*A) and an Adams-Novikov spectral sequence

$$E_{**}^2 = \text{Cotor}_{A_*A}(A_*, A_*X) \implies \pi_*X_{\hat{A}}.$$

If \mathcal{M}_A denotes the stack associated to the Hopf algebroid (A_*, A_*A) and F_X the graded sheaf associated with the comodule A_*X , this E^2 -term can be expressed as

$$E_{**}^2 = H^{**}(\mathcal{M}_A, F_X),$$

which is the cohomology of the stack \mathcal{M}_A with coefficients in the sheaf F_X .

In particular, if $f: A \rightarrow B$ is a morphism of MU -even ring spectra, we get a morphism of spectral sequences, and if f induces an equivalence of the associated stacks, then f induces an isomorphism of spectral sequences from the E^2 -term on. In particular, in this case, $X_{\hat{A}} \simeq X_{\hat{B}}$ if we can assure that the spectral sequences converge strongly. Note that we do not need an inverse map $B \rightarrow A$.

Theorem 3.3. *If $f: A \rightarrow B$ is a morphism of MU -even ring spectra inducing an equivalence of associated stacks, then $L_A \simeq L_B$.*

Proof. The argument outline above gives an almost-proof of this fact, but it puts us at the mercy of the convergence of the Adams-Novikov spectral sequences to the localizations $L_A X$ and $L_B X$. We give an argument that doesn't require such additional assumptions. Note that it is sufficient to show that $A_*X = 0$ if and only if $B_*X = 0$. Assume $A_*X = 0$. Then the A -based Adams-Novikov spectral sequence is 0 from E^1 on, thus the B -based Adams-Novikov spectral sequence is also trivial from E^2 on. This time, the spectral sequence converges strongly because it is conditionally convergent in the sense of Boardman [Boa99], which implies strong convergence if the derived E_∞ -term is 0 – but this is automatic since the E_r -terms are all trivial for $r \geq 0$. Thus $X_{\hat{B}}$ is contractible.

Now the Hurewicz map $X \rightarrow B \wedge X$ factors as $X \rightarrow L_B X \rightarrow X_{\hat{B}} \rightarrow B \wedge X$ by the universal property ① of the localization, since $X_{\hat{B}}$ is B -local. Thus $X \rightarrow B \wedge X$ is trivial. Using the ring spectrum structure on B , we see that $B \wedge X \rightarrow B \wedge B \wedge X \xrightarrow{\mu} B \wedge X$, which is the identity, is also trivial, so $B \wedge X \simeq *$. \square

In particular, this applies to the following cases:

Theorem 3.4. *We have*

$$L_{B(n)} \simeq L_{K(n)}$$

Let $I_n = (p, v_1, \dots, v_{n-1}) \triangleleft BP_$ and $E(k, n) = E(n)/I_k$ for $0 \leq k \leq n \leq \infty$. Then*

$$L_{v_k^{-1}E(k, n)} \simeq L_{K(k)}.$$

Proof. The first part is due to Ravenel [Rav84] and Johnson-Wilson [JW75], but they give a different proof without the Adams-Novikov spectral sequence.

To apply Theorem 3.3, it is useful to extend the ground ring of the homology theories in question from \mathbb{F}_p to \mathbb{F}_{p^n} , which does not change their localization functors. The Hopf algebroids for $B(n) \otimes \mathbb{F}_{p^n}$ and $K(n) \otimes \mathbb{F}_{p^n}$ both classify formal groups of height n . By Lazard's theorem, there is only one such group over \mathbb{F}_{p^n} up to isomorphism, which shows that the quotient map $B(n) \otimes \mathbb{F}_{p^n} \rightarrow K(n) \otimes \mathbb{F}_{p^n}$ induces an isomorphism of Hopf algebroids.

The second part works similarly by considering the maps of Hopf algebroids induced from

$$v_k^{-1}E(k, n) \leftarrow B(k)/(v_{n+1}, v_{n+2}, \dots) \rightarrow K(k)$$

which again all represent the stack of formal groups of height k . □

Theorem 3.5. *We have that*

$$L_{E(n)} \simeq L_{K(0) \vee K(1) \vee \dots \vee K(n)} \simeq L_{v_n^{-1}BP}.$$

Proof. With the notation of Theorem 3.4, since $E(n, n) = K(n)$ and $E(0, n) = E(n)$, it suffices to show that

$$L_{E(k, n)} \simeq L_{K(k) \vee E(k+1, n)}.$$

By Lemma 1.3, $L_{E(k, n)} \simeq L_{v_k^{-1}E(k, n) \vee E(k+1, n)}$. By Theorem 3.4, $L_{v_k^{-1}E(k, n)} \simeq L_{K(k)}$, and the result follows by induction.

The second equivalence can be proved by a similar argument, not needed here, and left to the reader. □

Theorem 3.6. *There is a homotopy pullback square*

$$\begin{array}{ccc} L_{K(1) \vee K(2)}X & \xrightarrow{\eta_{K(2)}} & L_{K(2)}X \\ \eta_{K(1)} \downarrow & & \downarrow \eta_{K(1)} \\ L_{K(1)}X & \xrightarrow{L_{K(1)}(\eta_{K(2)})} & L_{K(1)}L_{K(2)}X \end{array}$$

Proof. This is an application of Prop. 2.2. We need to see that $K(2)_*(L_{K(1)}X) = 0$ for any X . To see this, let $\alpha: \Sigma^k M(\mathbb{Z}/p) \rightarrow M(\mathbb{Z}/p)$ be the Adams map, which induces multiplication with a power of v_1 in $K(1)$ and is trivial in $K(2)$. Here $k = 2p - 2$ for odd p and $k = 8$ for $p = 2$.

Let X be $K(1)$ -local. Then so is $X \wedge M(\mathbb{Z}/p)$, and since $\Sigma^k X \wedge M(\mathbb{Z}/p) \xrightarrow{\alpha} X \wedge M(\mathbb{Z}/p)$ is a $K(1)$ -isomorphism, it is a homotopy equivalence. On the other hand, $\alpha_*: K(2)_*(\Sigma^k X \wedge M(\mathbb{Z}/p)) \rightarrow K(2)_*(X \wedge M(\mathbb{Z}/p))$ is trivial, thus $K(2)_*(X \wedge M(\mathbb{Z}/p)) = 0$. By the Künneth isomorphism, $K(2)_*(X) = 0$. □

The same result holds true for any $K(m)$ and $K(n)$ with $m < n$; $M(\mathbb{Z}/p)$ and α then have to be replaced by a type- m complex and its v_m -self map in the argument. We briefly recall some basic facts around the periodicity theorem.

Definition 3.7. A finite p -local CW -spectrum X has *type* n if $K(n)_*(X) \neq 0$ but $K(k)_*(X) = 0$ for $k < n$. For example, the sphere has type 0, the Moore spectrum $M(\mathbb{Z}/p)$ has type 1, and the the cofiber of the Adams map has type 2.

Theorem 3.8 ([DHS88, HS98]). *Every type- n spectrum X admits a v_n -self map, i. e. a map $f: \Sigma^n X \rightarrow X$ which induces multiplication by a power of v_n in $K(n)_*(X)$.*

The periodicity theorem implies that there exist type- n complexes for every $n \in \mathbb{N}$. They can be constructed iteratively, starting with the sphere, by taking cofibers of v_k -self maps. Thus, there exist multi-indices $I = (i_0, \dots, i_{n-1})$ and spectra $\mathbb{S}^0/(v^I)$ such that $BP_*(\mathbb{S}^0/(v^I)) = BP_*(v^I)$, where $(v^I) = (p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$. These are sometimes called *generalized Moore spectra*. It is an open question what the minimal values of I are (they certainly depend on the prime.)

4. THE HASSE SQUARE

In this section, we will study algebraic interpretations of $K(n)$ -localization in terms of formal groups and elliptic curves.

Proposition 4.1. *Let E be a complex oriented ring spectrum over $\mathbb{S}_{(p)}$ and define*

$$E' = \operatorname{holim}_{(i_0, \dots, i_{n-1}) \in \mathbb{N}^n} v_n^{-1} E / (p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}}).$$

Here v_i are the images of the classes in BP of the same name under the orientation $BP \rightarrow E$. Then $L_{K(n)} E \simeq E'$.

Proof. We also denote by $I_n \triangleleft E_*$ the image of the ideal of the same name in BP . As $v_n^{-1} E / I_n$ is a $B(n)$ -module spectrum, it is $B(n)$ -local by Lemma 1.2, thus by Theorem 3.4 also $K(n)$ -local. Each spectrum $v_n^{-1} E / (v^I)$ (using multi-index notation) is constructed from $v_n^{-1} E / I$ by a finite number of cofibration sequences, thus it is also $K(n)$ -local. Since homotopy limits of local spectra are again local (Lemma 1.4), E' is $K(n)$ -local, and it remains to show that $K(n)_*(E) \cong K(n)_*(E')$. The coefficient rings of the Morava K -theories $K(n)$ are graded fields, hence they have Künneth isomorphisms. Thus it suffices to show that $E \wedge X \rightarrow E' \wedge X$ is a $K(n)$ -equivalence for some X with nontrivial $K(n)_*(X)$. Choose $X = \mathbb{S}^0 / (v^J)$ to be a generalized Moore spectrum of type n , for some multi-index J . Then

$$E' \wedge X \simeq \operatorname{holim}_{I \in \mathbb{N}^n} (v_n^{-1} E / (v^I) \wedge \mathbb{S}^0 / (v^J)) \simeq v_n^{-1} E / (v^J).$$

Thus $K(n)_*(E \wedge X) = K(n)_*(E / v^J) = K(n)_*(v_n^{-1} E / v^J) = K(n)_*(E' \wedge X)$. \square

Now we will specialize to an elliptic spectrum E defined over the ring E_0 with associated elliptic curve C_E over $\operatorname{Spec} E_0$. Proposition 4.1 in particular tells us that

$$\pi_0 L_{K(1)} E \cong \lim_i v_1^{-1} E_0 / (p^i),$$

which is the ring of functions on $\operatorname{Spf}((E_0)_p^\wedge)^{\operatorname{ord}}$, the ordinary locus of the formal completion of $\operatorname{Spec} E_0$ at p , i.e. the sub-formal scheme over which C_E is ordinary. In particular, if E_0 is an \mathbb{F}_p -algebra, $\pi_0 L_{K(1)} E \cong v_1^{-1} E_0$ is just the (non-formal) ordinary locus of E_0 . Similarly,

$$\pi_0 L_{K(2)} E \cong \lim_{i_0, i_1} v_2^{-1} E_0 / (p^{i_0}, v_1^{i_1}) = \lim_{i_0, i_1} E_0 / (p^{i_0}, v_1^{i_1})$$

is the ring of functions on the formal completion of $\operatorname{Spec} E_0$ at the supersingular locus at p . The last equality holds because any elliptic curve has height either 1 or 2 over \mathbb{F}_p , thus v_2 is a unit in $E_0 / (p, v_1)$ and hence in $E_0 / (p^{i_0}, v_1^{i_1})$.

Lemma 4.2. *Any p -local elliptic spectrum E is $E(2)$ -local.*

Proof. We need to show that for any W with $E(2)_* W = 0$, we have that $E_* W = 0$. By Theorems 3.4 and 3.5, this is equivalent to $B(i) = 0$ for $0 \leq i \leq 2$. That is,

$$\begin{aligned} p^{-1} BP \wedge W &\simeq * \\ v_1^{-1} BP / p \wedge W &\simeq * \\ v_2^{-1} BP / (p, v_1) \wedge W &\simeq *. \end{aligned}$$

Now since E is a BP -ring spectrum, the same equalities hold with BP replaced by E . It follows from Lemma 1.3 that

$$\begin{aligned} E / (p, v_1) \wedge W &\simeq v_2^{-1} E / (p, v_1) \wedge W \simeq * \text{ and } v_1^{-1} E / p \wedge W \simeq * \Rightarrow E / p \wedge W \simeq * \\ E / p \wedge W &\simeq * \text{ and } p^{-1} E \wedge W \simeq * \Rightarrow E \wedge W \simeq *. \end{aligned}$$

\square

Corollary 4.3 (the “Hasse square”). *For any elliptic spectrum E , there is a pullback square*

$$\begin{array}{ccc} E_p^\wedge & \longrightarrow & L_{K(2)} E \\ \downarrow & & \downarrow \\ L_{K(1)} E & \longrightarrow & L_{K(1)} L_{K(2)} E. \end{array}$$

Proof. It follows from Lemma 3.6 that the pullback is $L_{K(1) \vee K(2)} E$. Now consider the arithmetic square

$$\begin{array}{ccc} L_{K(0) \vee K(1) \vee K(2)} E & \longrightarrow & L_{K(1) \vee K(2)} E \\ \downarrow & & \downarrow \\ L_{K(0)} E & \longrightarrow & L_{K(0)} L_{K(1) \vee K(2)} E. \end{array}$$

Since $L_p L_{K(0)} X = L_p L_{\mathbb{Q}} X = *$, applying the p -completion functor L_p , we see that top horizontal map

$$L_p E \simeq L_p L_{K(0) \vee K(1) \vee K(2)} E \rightarrow L_p L_{K(1) \vee K(2)} E \simeq L_{K(1) \vee K(2)} E$$

is an equivalence, hence the result. \square

REFERENCES

- [Ada74] J. F. Adams. *Stable homotopy and generalised homology*. University of Chicago Press, Chicago, Ill., 1974. Chicago Lectures in Mathematics.
- [BL10] Tilman Bauer and Assaf Libman. A_∞ -monads and completion. *Journal of Homotopy and Related Structures*, 5(1):133–155, 2010.
- [Boa99] J. Michael Boardman. Conditionally convergent spectral sequences. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 49–84. Amer. Math. Soc., Providence, RI, 1999.
- [Bou75] A. K. Bousfield. The localization of spaces with respect to homology. *Topology*, 14:133–150, 1975.
- [Bou79] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.
- [DHS88] Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith. Nilpotence and stable homotopy theory. I. *Ann. of Math. (2)*, 128(2):207–241, 1988.
- [EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.
- [HS98] Michael J. Hopkins and Jeffrey H. Smith. Nilpotence and stable homotopy theory. II. *Ann. of Math. (2)*, 148(1):1–49, 1998.
- [JW75] David Copeland Johnson and W. Stephen Wilson. BP operations and Morava’s extraordinary K -theories. *Math. Z.*, 144(1):55–75, 1975.
- [Rav84] Douglas C. Ravenel. Localization with respect to certain periodic homology theories. *Amer. J. Math.*, 106(2):351–414, 1984.
- [Sul05] Dennis P. Sullivan. *Geometric topology: localization, periodicity and Galois symmetry*, volume 8 of *K-Monographs in Mathematics*. Springer, Dordrecht, 2005. The 1970 MIT notes, Edited and with a preface by Andrew Ranicki.