HOPKINS TALK #1: STACKS AND COMPLEX ORIENTED COHOMOLOGY THEORIES

NOTES BY: MICHAEL A. HILL

1. Introductory Remarks

This is the first in a series of lectures given by Mike Hopkins at the Talbot workshop in April of 2007. For this talk, Mike was asked to talk about "the *tmf* theorem" and to explain some of the motivation and intuition behind the stacky language. As those who have heard him speak know, Mike is a very entertaining lecturer. Sadly, these notes reflect only the mathematical content of his talks, leaving the many funny comments and laughter in the memories of the audience. All of the content that follows is taken essentially directly from Mike's talk. We have expanded very slightly some portions and we have reorganized some of the material, and we take full responsibility for any mathematical or typographical errors.

2. The "*tmf* Theorem" and MU

2.1. Main Theorem.

Theorem 2.1. *There exists a "unique" sheaf $\mathcal{O}^{top}$ of E_∞-ring spectra on $\mathcal{M}_{ell}^{ét}$ such that for $U = \operatorname{Spec} R \to \mathcal{M}_{ell}$, one has that $\mathcal{O}^{top}(U)$ is the Landweber exact theory corresponding to the composite $U \to \mathcal{M}_{ell} \to \mathcal{M}_{FG}$.*

The argument for the existence of such a sheaf is via obstruction theory: one shows that such a sheaf cannot *not* exist. Moreover, the produced object has automorphisms, so in some sense it does not uniquely pin down *tmf*. Lurie's construction using derived elliptic curves achieves this.

This note will focus on some of the back-ground and motivation for the problem. In particular, we will have a broad overview for why the language of stacks, language which was relatively unused even in algebraic geometry around the time *tmf* was first constructed, plays such a prominent role.

2.2. Complex Oriented Cohomology Theories.

Classically, MU was understood through the relationship with formal group laws:

$$\{\text{Complex oriented cohomology theories}\} \leftrightarrow \{\text{Formal Group Laws}\}.$$

Beginning with Quillen's landmark result that MU carries the universal formal group law, this concept informed most of the results in stable homotopy theory. The triumph of Miller-Ravenel-Wilson, Landweber, and many mathematicians of that generation was finding good ways to coordinatize. They used this to successfully carry out incredible computations, revealing many subtleties of the stable homotopy category. For later generations, it was impossible to "out-compute" these mathematicians, so they sought more conceptual reasons for the results. This lead naturally to looking in a coordinate free context, focusing on formal groups rather than formal group laws, and this shift lead naturally to consider sheaves of spectra and associated moduli problems like the one for *tmf*.

Date: April 29, 2009.

2.3. **Ravenel's Filtration of MU.** Given a ring spectrum R, we can identify multiplicative maps $MU \to R$ with coordinates on the formal group over R. We know that MU is the Thom spectrum of the canonical bundle over BU, and Ravenel gave an important filtration of MU that played roles in the Nilpotence and Periodicity theorems. These also play a key role in Ravenel's approach to computing the stable homotopy groups of spheres, and in general, their homotopy groups are essentially as complicated as those of the sphere.

Bott periodicity shows that $BU = \Omega SU$. If we filter SU by the Lie groups $SU(n)$, then we get a filtration

$$\{*\} = \Omega SU(1) \subset \cdots \subset \Omega SU(n) \subset \cdots \subset \Omega SU = BU.$$

Passing to Thom spectra produces a filtration of MU of the form

$$S^0 \to \cdots \to X(n) \to \cdots \to MU.$$

The spectra $X(n)$ are actually homotopy commutative (E_2) ring spectra, since the map $\Omega SU(n) \to \Omega SU$ is a 2-fold loop map. This is a refinement of an easier result.

Proposition 2.2. *If S is a group and $\zeta\colon S \to BU$ is a loop-map, then S^ζ is a ring spectrum.*

Proof. We have a commutative square

$$\begin{array}{ccc} S \times S & \xrightarrow{\zeta\times\zeta} & BU \times BU \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \oplus} \\ S & \xrightarrow[\zeta]{} & BU \end{array}$$

which gives us two ways to describe Thom spectrum over $S \times S$. Naturality of the Thom spectrum produces a map

$$(S \times S)^{\zeta\circ\mu} \to S^\zeta.$$

However, since $\zeta \circ \mu = \zeta \oplus \zeta$, we learn that

$$(S \times S)^{\zeta\circ\mu} = (S \times S)^{\zeta\oplus\zeta} = S^\zeta \wedge S^\zeta.$$

Thus we get that S^ζ is a ring spectrum. □

While these spectra are *not* complex orientable, they filter the story of multiplicative maps from MU.

If R is complex orientable, then the Thom isomorphism shows that

$$R_*MU \cong R_*BU,$$

and this in turn is the symmetric algebra over R_* of $R_*\mathbb{C}P^\infty$. Since R is complex orientable, we also know that

$$MultMaps(MU, R) \leftrightarrow Ring(R_*MU, R_*),$$

and the above analysis shows that this is determined by an R_*-module homomorphism from $R_*\mathbb{C}P^\infty$ to R_*. We will use this to help us understand maps from $X(n)$ to R, but we must first refine this geometrically.

There is a natural map $\mathbb{C}P^\infty \to BU$ classifying $L - 1$, where L is the canonical line bundle (the -1 arises since maps into BU classifies virtual bundles of virtual dimension 0). Passing to Thom spectra we get a map

$$\Sigma^{-2}\mathbb{C}P^\infty \to (\mathbb{C}P^\infty)^{(L-1)} \to MU.$$

This realizes the copy of $R_*\mathbb{C}P^\infty$ we see in R_*MU. We also see that the cells in $\mathbb{C}P^\infty$ give rise to the terms in the formal power series expansion of a coordinate for the formal group.

To build in the $X(n)$ spectra, we isolate the copy of $\mathbb{C}P^n$ inside $\mathbb{C}P^\infty$. We start by building a map $S^1 \times \mathbb{C}P^{n-1} \to SU(n)$. Let $r\colon S^1 \times \mathbb{C}P^{n-1} \to U(n)$ be the map which associates to a pair (λ, ℓ) the rotation of $\mathbb{C}^n$ about the line ℓ with angle of rotation λ. If we let ℓ_0 correspond to the base point of $\mathbb{C}P^{n-1}$, then we can make r into a map to $SU(n)$ by dividing by the value at ℓ_0:

$r(\lambda,\ell)/r(\lambda,\ell_0)$. The points $S^1\times *$ and $*\times\mathbb{C}P^{n-1}$ all go to the base-point 1 in $SU(n)$, we this map descends to a map

$$S^1\times\mathbb{C}P^{n-1}\to SU(n).$$

Taking adjoints gives us a map $\mathbb{C}P^{n-1}\to\Omega SU(n)$, and passing to Thom spectra gives us a natural map

$$\Sigma^{-2}\mathbb{C}P^n\cong(\mathbb{C}P^{n-1})^{L-1}\to X(n).$$

Using an analysis similar to that above for MU, we the conclude that multiplicative maps from $X(n)$ to R are in one-to-one correspondence with coordinates modulo degree n on the formal group.

Remark 2.3. If R is not complex orientable, then we can still make sense of multiplicative maps from $X(n)$ to R. These are sometimes called "formal group law chunks".

Even though the homotopy groups of $X(n)$ are not known for any finite values of n, we do know that these spectra are "flat":

$$X(n)\wedge X(n)=X(n)[b_1,\dots,b_{n-1}],$$

where $|b_i|=2i$, and where we interpret the right-hand side as being a wedge over all monomials in the graded ring $Z[b_1,\dots b_{n-1}]$. This means that although the homotopy groups are bad, the pair $\big(X(n)_*,X(n)_*X(n)\big)$ forms a Hopf algebroid. In particular, we can hope to understand algebraically the $X(n)$-based Adams Novikov spectral sequence.

3. Stacks from Spectra

If X is any spectrum, then we can naturally associate a sheaf over $\mathcal{M}_{FG}$ using the MU-based Adams resolution:

$$MU_*X\rightrightarrows MU_*MU\otimes_{MU_*}MU_*X\dots.$$

We can analyze this in pieces. The first part is an MU_*-module, so it therefore corresponds to a sheaf over $Spec(L)$, the prime spectrum of the Lazard ring. Similar, the second term is a sheaf on $Spec(L)\times_{\mathcal{M}_{FG}}Spec(L)$. The left and right units are exactly the start of the structure maps giving descent data, and in fact, this algebraic Adams resolution produces exactly a sheaf $\mathcal{M}_X$ over $\mathcal{M}_{FG}$. If X is a ring spectrum, then $\mathcal{M}_X$ will actually be a stack over $\mathcal{M}_{FG}$. In this way, we assign to ring spectra nice stacks over $\mathcal{M}_{FG}$.

There are several important examples:

(1) $\mathcal{M}_{S^0}=\mathcal{M}_{FG}$, essentially by definition.
(2) $\mathcal{M}_{X(n)}$ is the stack representing formal groups together with a coordinate modulo degree n.

In good cases, if X and Y are ring spectra and if they are nice, then $\mathcal{M}_{X\wedge Y}$ is the stacky pull-back

$$\begin{array}{ccc}\mathcal{M}_{X\wedge Y} & \longrightarrow & \mathcal{M}_X\\ \downarrow & & \downarrow\\ \mathcal{M}_Y & \longrightarrow & \mathcal{M}_{FG}\end{array}\tag{1}$$

and this allows us to actually do some key computations.

We'd like to show this via descent. We need to know that $MU_*(X\wedge Y)\cong MU_*(X)\otimes_{MU_*}MU_*(Y)$. If either $MU_*(X)$ or $MU_*(Y)$ is flat, then the Künneth spectral sequence for MU collapses, giving us the desired isomorphism. In this case, when we form the resolution for $X\wedge Y$, we see that it is just the tensor product of the resolutions for X and for Y.

If R is complex orientable, then the moduli stack $\mathcal{M}_R$ is actually a scheme. In this case, the Adams-Novikov resolution collapses, so $\mathcal{M}_R=\operatorname{Spec}\pi_*R$ with no stackiness. We can therefore view stackiness in $\mathcal{M}_X$ as some sort of measure of the failure of complex orientability.

3.1. **Usefulness of Stacks: Calculating with $\mathcal{M}_X$.** We have produced for any ring spectrum X a moduli stack $\mathcal{M}_X$ over $\mathcal{M}_{FG}$. This arose as the stack associated to the Hopf algebroid $(MU_*X, MU_*MU \otimes MU_*X)$. The important part is that at the end of the day we care only about the underlying stack, not the particular choice of presentation. This gives us lots of flexibility!

We need to figure out ways to build a simpler presentation, and in many cases, the spectra $X(n)$ will help this along. First a fact: if R is a nice ring spectrum, then we can use R instead of MU and get an equivalent stack, looking at

$$R_*X \rightrightarrows (R \wedge R)_*X \ldots.$$

What does "nice" mean here?

(1) $R \wedge MU$ is complex orientable.
(2) R_*R is a flat R_*-module (this guarantees that (R_*, R_*R) is a Hopf algebroid and that the resolution is descent data).
(3) R has "full support": X is R-local (this ensures that topological resolution

$$R \wedge X \rightrightarrows R \wedge R \wedge X \ldots$$

is a resolution of X.)

The fact that these approaches give the same underlying stack is a reformulation to many of the classical change-of-rings theorems used for computations with the Adams spectral sequence. We will use this approach to facilitate some computations.

4. Two Important Examples

4.1. **The $X(5)$-homology of tmf.** This computation will tie the Weierstrass Hopf algebroid firmly to tmf for topological reasons, and it will allow use to see the usefulness of the stacky pullback square Equation**??**. We first recall two things:

(1) $\mathcal{M}_{X(5)}$ is the moduli stack of formal groups together with a coordinate modulo order 5.
(2) $\mathcal{M}_{tmf}$ is the Weierstrass moduli stack $\overline{\mathcal{M}}^+_{ell}$.

The pullback square then says that

$$\begin{array}{ccc} \mathcal{M}_{tmf\wedge X(5)} & \longrightarrow & \overline{\mathcal{M}}^+_{ell} \\ \downarrow & & \downarrow \\ \mathcal{M}_{X(5)} & \longrightarrow & \mathcal{M}_{FG}. \end{array}$$

The data given by $\mathcal{M}_{tmf\wedge X(5)}$ is then an elliptic curve C together with a coordinate modulo degree 5 on its formal group. We saw in an earlier talk that this is the same data as giving C and a coordinate of degree 5 on C, which is exactly a way to identify the Weierstrass equation of C. We therefore learn that $\mathcal{M}_{tmf\wedge X(5)}$ is affine:

$$\mathcal{M}_{tmf\wedge X(5)} = \operatorname{Spec} \mathbb{Z}[a_1, a_2, a_3, a_4, a_6],$$

so

$$\pi_* tmf \wedge X(5) = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6].$$

Running a similar analysis for $tmf \wedge X(5) \wedge X(5)$ shows that the $X(5)$-based Adams resolution of tmf gives exactly the cobar resolution of the Weierstrass Hopf algebroid!

4.2. **The Stack $\mathcal{M}_{bo}$.** Here we shift focus from periodic, complex orientable spectra to connective (possibly non-complex orientable) ones. In particular, we will look at the spectrum bo, the (-1)-connected cover of KO. This spectrum has been extremely useful in stable homotopy, as computations with it tend to be quite tractable.

Recall from Bott periodicity that the spaces in the spectrum for KO are 8-fold periodic, starting with the space $\mathbb{Z} \times BO$. For the spectrum bo, there are no spaces in the negative degrees for the spectrum, while in degree $8k$, we have the $8k$-connected cover of BO, $BO\langle 8k\rangle$, and the other spaces are what they have to be. This restriction of the homotopy groups of the spaces in the spectrum ensures that there are no negative homotopy groups for bo. There is similarly a spectrum bu for complex connective K-theory.

Proposition 4.1. *We have an equivalence*

$$bo \wedge \Sigma^{-2}\mathbb{C}P^2 \simeq bu.$$

This proposition is actually an interpretation of the Bott fibrations. Thus, although bo is not complex orientable, the spectrum $bo \wedge \Sigma^{-2}\mathbb{C}P^2$ *is*. Thus we can calculate the stack $\mathcal{M}_{bo}$ from this and from the scheme $\mathcal{M}_{bu}$.

The big problem is that $\Sigma^{-2}\mathbb{C}P^2$ is not a ring spectrum. We would really like to use it, though, since it has only two cells. Thus we should be able to realize our stack $\mathcal{M}_{bo}$ as being double covered by a scheme ($\mathcal{M}_{bu}$ in some stacky way). However, our analysis of the spectra $X(n)$ shows that $\Sigma^{-2}\mathbb{C}P^2$ embeds in a nice ring spectrum: $X(2)$.

The Thom isomorphism shows that

$$H_*X(2) = H_*\Omega SU(2) = \mathbb{Z}[b_1],$$

where b_1 is in degree 2 and corresponds to $\mathbb{C}P^2$.

Remark 4.2. More is true in general. The homology of all of the spaces $X(n)$ is a polynomial algebra:

$$H_*X(n) = \mathbb{Z}[b_1, \ldots, b_{n-1}],$$

where the classes b_i here are essentially the same as those occurring in $X(n) \wedge X(n)$.

In fact, we know a little more for $X(2)$.

Proposition 4.3. *There is a Thom spectrum Y_4 such that*

$$H_*Y_4 = \mathbb{Z}[c], \quad |c| = 4,$$

and $X(2) = \Sigma^{-2}CP^2 \wedge Y_4$.

This in turn implies that while we cannot hope to understand $\pi_*X(2)$, we can actually understand every piece of the resolution

$$X(2)_*bo \rightrightarrows X(2)_*X(2) \otimes_{X(2)_*} X(2)_*bo \ldots.$$

Corollary 4.4. *As rings*

$$X(2)_*bo = \mathbb{Z}[b, c], \quad |b| = 2,$$

and

$$X(2)_*X(2) \otimes_{X(2)_*} X(2)_*bo = \mathbb{Z}[b, c, r], \quad |r| = 2.$$

The last equality comes from the isomorphism $X(2)_*X(2) = X(2)_*[b_1]$, and we have renamed b_1 to r.

Since everything in sight is torsion free, we can compute the right units by embedding the computation in the rationalization or by pushing down to MU. In any case, we see that the Hopf algebroid $\big(X(2)_*bo, X(2)_*X(2) \otimes_{X(2)_*} X(2)_*bo\big)$ corepresents the following data: Curves of the form $y = x^2 + bx + c$, together with coordinate changes $x \mapsto x + r$. In other words, the right unit in the Hopf algebroid is given by

$$\begin{pmatrix} b \\ c \end{pmatrix} \mapsto \begin{pmatrix} b + 2r \\ c + br + r^2 \end{pmatrix}.$$

If we write our curve instead as $y^{p-1} = x^p + bx + c$, then we see that this curve plays the role at 2 for elliptic curves at $p = 3$.

This Hopf algebroid already gives a simpler presentation of the stack:

$$\mathcal{M}_{bo} = Stack(\mathbb{Z}[b, c], \mathbb{Z}[b, c, r]).$$

This is still a little flabby. We should have something of degree 2, reflecting the fact that $\Sigma^{-2}\mathbb{C}P^2$ had only 2 cells. Here finding yet another presentation is helpful. We can do this by thinking about what a point on the stack means. To give $\operatorname{Spec} A \to \mathcal{M}_{bo}$ is to give a faithfully flat extension $A \to B$, together with a curve $y = x^2 + bx + c$ over B and with two extensions over $B \otimes_A B$ together with a morphism between them. In other words, our stack satisfies faithfully flat descent, so we can realize maps from $\operatorname{Spec} A$ to $\mathcal{M}_{bo}$ by faithfully flat covers together with descent data.

Let us apply this with $\mathbb{Z}[b]$ instead of $\mathbb{Z}[b, c]$. There is a map $\operatorname{Spec} \mathbb{Z}[b] \to \mathcal{M}_{bo}$ representing the curve $y = x^2 + bx$. This admits coordinate transformations of the form $x \mapsto x + r$ provided $r^2 + br = 0$. In fact, this map is a flat cover of $\mathcal{M}_{bo}$. We check this in two steps.

We can check that the map is flat by checking pulling back along the affine cover of $\mathcal{M}_{bo}$ by $\operatorname{Spec} \mathbb{Z}[b, c]$ representing the universal curve $y = x^2 + bx + c$:

$$\begin{array}{ccc} \operatorname{Spec} \mathbb{Z}[b,c,r]/r^2 + br + c & \longrightarrow & \operatorname{Spec} \mathbb{Z}[b] \\ {\scriptstyle \binom{b}{c} \mapsto \binom{b+2r}{0}} \downarrow & & \downarrow {\scriptstyle y = x^2 + bx} \\ \operatorname{Spec} \mathbb{Z}[b,c] & \xrightarrow[y = x^2 + bx + c]{} & \mathcal{M}_{bo} \end{array}$$

The stacky pullback is represented by pairs of curves $(y = x^2 + b'x, y = x^2 + bx + c)$, together with a coordinate transformation that identifies these. This amounts to translating one of the roots of $x^2 + bx + c$ to 0, so the pullback is represented by $\operatorname{Spec} \mathbb{Z}[b, c, r]/r^2 + br + c$. This is flat over $\operatorname{Spec} \mathbb{Z}[b, c]$.

We also see immediately that this is a cover. If we have a curve $y = x^2 + bx + c$, then we can put it into the form $y = x^2 + b'x$ by first adjoining a root of $x^2 + bx + c$ and then translating by this root. Adding in a root of a monic polynomial produces a faithfully flat extension, so we see that every isomorphism class of curves over $\operatorname{Spec} A$ has a representative with $c = 0$.

Combining these two results, we see that there is a simpler presentation of the stack. It is the stack associated to the Hopf algebroid

$$(\mathbb{Z}[b], \mathbb{Z}[b, r]/r^2 + 2r), \quad b \mapsto b + 2r.$$

In particular, we see that the map $\operatorname{Spec} \mathbb{Z}[b] \to \mathcal{M}_{bo}$ is a double cover, which is exactly what we wanted!

This example is an important toy example for understanding how *tmf* works. At $p = 2$, we can find an 8-fold cover of the moduli stack (or a full 24-fold cover if we impose a level 3 structure), and this corresponds to seeing that there is an 8-cell complex X such that $tmf \wedge X$ is Landweber exact. Similarly, at $p = 3$, we can find a 3-fold cover, corresponding to a 3-cell complex Y such that $tmf \wedge Y$ is also Landweber exact. These are very important for carrying out the computations with *tmf*!