LECTURE BY MICHAEL HOPKINS: FROM SPECTRA TO STACKS

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This talk aims to explain why the language of stacks, language which was relatively unused even in algebraic geometry around the time *tmf* was constructed, plays such a prominent role in the theorem expressing the existence of *TMF*.

Let me try to explain where it all came from... I come from an era in homotopy theory when all that anybody did was to compute things. Doug Ravenel and others tried to make sense of all these computations, but it was only the people in my generation who started to think about how to understand these computations in a more conceptual way. Nevertheless, I still tend to see everything through a computational lens.

For me, TMF came out of what I call "designer spectra" or "designer homotopy types". The first examples of such designer objects are the Eilenberg-MacLane spaces: they are spaces designed to have certain homotopy theoretic properties, but with otherwise not much geometry. The appearance of those homotopy types marked the point when people started to make actual progress, e.g., Serre's first computations of homotopy groups of spheres. Before that, just using geometric methods, all that one could prove (using simplicial approximation) was that these homotopy groups are countable. From then on, every advance in homotpy theory was spearheaded by the appearance of some special designer objects: spectra, various forms of K-theory, the Adams spectral sequence, Brown-Gitler spectra, etc.

One of the things about TMF that we were really proud of is that, suddenly, this was something that we could explain in a language that did not involve any corky, weird homotopy types, so that someone who did not know too much about homotopy theory could come in and do stuff with TMF — for example, someone like Jacob [laugh from the audience].

Even tough the TMF story might make more sense from the point of view of elliptic curves, I am going to tell you today about some of those older designer objects, and I hope that they will help fit the story together.

Let us start by recalling the main theorem:

Theorem 0.1. *There exists a unique sheaf $\mathcal{O}^{top}$ of E_∞-ring spectra on $(\mathcal{M}_{ell})_{\text{ét}}$ such that for an étale map $f : \operatorname{Spec} R \to \mathcal{M}_{ell}$, one has that $\mathcal{O}^{top}(\operatorname{Spec} R)$ is the Landweber exact theory corresponding to the composite $\operatorname{Spec} R \to \mathcal{M}_{ell} \to \mathcal{M}_{FG}$.*

To be more precise, the spectrum $\mathcal{O}^{top}(\operatorname{Spec} R)$ is even and periodic, its π_0 is identified with R, and the formal group corresponding to it is identified with the formal completion of the elliptic curve classified by f.

The argument for the existence of such a sheaf is via obstruction theory: instead of showing that the sheaf exists, one shows that it cannot *not* exist. Though the resulting sheaf is unique up to isomorphism, it has automorphisms, and so it fails to be unique up to unique isomorphism. It is therefore only unique in a weak sense. It is however possible to uniquely specify such a sheaf, and this was accomplished by Jacob Lurie's construction using derived elliptic curves. We should also note that the initial version of the theorem only produced a sheaf of A_∞-ring spectra, and it was Jacob Lurie who pointed out that $\mathcal{O}^{top}$ is actually a sheaf of E_∞-ring spectra. Later on, the obstruction theory could be adapted to prove that stronger result, but this was not part of the original approach.

Since Quillen's landmark result that MU carries the universal formal group law, the correspondence between complex-oriented cohomology theories and formal group laws has informed most of the results in stable homotopy theory. The triumph of Miller-Ravenel-Wilson, Landweber, and many mathematicians of that generation was finding good ways to coordinatize the formal groups and their associated cohomology theories. They used these coordinates to successfully carry out incredible computations, revealing many subtleties of the stable homotopy category. For the later generations, it was impossible to "out-compute" the earlier mathematicians, so people sought more conceptual formulations of what was known. This lead naturally to looking in a coordinate-free context, focusing on formal groups rather than formal group laws, and the culmination of that point of view was to realize that one should really be thinking about the moduli stack $\mathcal{M}_{FG}$.

1. Ravenel's Filtration of MU

Given a ring spectrum R, we can identify multiplicative maps $MU \to R$, that is, maps of ring spectra (not to be confused with A_∞ or E_∞ maps), with coordinates on the formal group associated to R. If there is a multiplicative map at all then there is a formal group law on R, and any two multiplicative maps define isomorphic formal groups. The formal group is therefore intrinsic to R, and it exists if and only if there exits a multiplicative map from MU into it, that is, if and only if R is complex orientable. For example, if a multiplicative cohomology theory is even and periodic, then there exists such a map, which means that there exists a formal group associated to R. There is however no preferred map $MU \to R$, and so the formal group does not come with any preferred choice of coordinate.

If R is complex orientable, then, by the Thom isomorphism, there is an isomorphism of R_*-algebras

$$R_*MU \;\cong\; R_*BU.$$

The latter is in turn the symmetric algebra over R_* of $R_*\mathbb{C}P^\infty$ (the generator of $R_0\mathbb{C}P^\infty$ being identified with the multiplicative unit). Since R is complex orientable, we also know that

$$Hom^{\otimes}(MU, R) \;\cong\; \mathrm{Alg}(R_*MU, R_*),$$

where $Hom^{\otimes}$ denotes maps of ring spectra and Alg denotes R_*-algebra maps. A multiplicative map $MU \to R$ is therefore equivalent to an R_*-module homomorphism from $R_*\mathbb{C}P^\infty$ (reduced homology) to R_*. There is also a geometric way of stating the above facts. Recall that there is an important map $\mathbb{C}P^\infty \to BU$ that classifies the virtual vector bundle $L-1$, where L is the canonical line bundle, and 1 the trivial complex line bundle. Passing to Thom spectra, we get a map

$$\Sigma^{-2}\mathbb{C}P^\infty \cong (\mathbb{C}P^\infty)^{(L-1)} \to MU$$

that realizes the copy of $R_*\mathbb{C}P^\infty$ that we see inside R_*MU (for $n \geq 0$, the generator of $R_{2n}\mathbb{C}P^\infty$ corresponds to the $2n$-cell of $\Sigma^{-2}\mathbb{C}P^\infty$). The cells of $\Sigma^{-2}\mathbb{C}P^\infty$ therefore correspond to the terms in the formal power series expansion of the coordinate on the formal group.

Ravenel introduced an important filtration $\{X(n)\}_{n\geq 1}$ of MU, which played a key role in the nilpotence and periodicity theorems, and also in his approach to computing the stable homotopy groups of spheres. It is defined as follows. First of all, by Bott periodicity, we know that $BU = \Omega SU$. If we filter SU by the Lie groups $SU(n)$, then we get a filtration

$$\{*\} = \Omega SU(1) \subset \cdots \subset \Omega SU(n) \subset \cdots \subset \Omega SU = BU.$$

Recall that MU is the Thom spectrum of the universal virtual bundle over BU. Applying the Thom spectrum construction to the above filtration of BU produces a filtration

$$\mathbb{S}^0 = X(1) \to X(2) \to \cdots \to X(n) \to \cdots \to MU$$

of MU. The spectra $X(n)$ are homotopy commutative and their homotopy groups are roughly as complicated as those of the sphere. They are actually E_2-ring spectra because $\Omega SU(n) \to \Omega SU = BU$ is a 2-fold loop map. More generally, the Thom spectrum of a k-fold loop map is always an E_k-ring spectrum. For example:

Proposition 1.1. *If S is an H-space and $\zeta\colon S \to BU$ is an H-map, then S^ζ is a ring spectrum.*

Proof. We have a commutative square

$$\begin{tikzcd} S \times S \arrow[r, "\zeta\times\zeta"] \arrow[d, "\mu"'] & BU \times BU \arrow[d, "\oplus"] \\ S \arrow[r, "\zeta"'] & BU \end{tikzcd}$$

which gives us two ways to describe Thom spectrum over $S \times S$. Naturality of the Thom spectrum produces a map

$$(S \times S)^{\zeta\circ\mu} \to S^{\zeta}.$$

However, since $\zeta \circ \mu = \zeta \oplus \zeta$, we learn that

$$(S \times S)^{\zeta\circ\mu} = (S \times S)^{\zeta\oplus\zeta} = S^{\zeta} \wedge S^{\zeta}.$$

Thus we have that S^{ζ} is a ring spectrum. □

Although the spectra $X(n)$ are not complex orientable (only MU is), there is a similar story that involves them, and that has something to do with complex orientations and formal groups. We first isolate the copy of $\mathbb{C}P^{n-1}$ inside $\mathbb{C}P^{\infty}$, and build a map $S^1 \times \mathbb{C}P^{n-1} \to SU(n)$, as follows. Let $r\colon S^1 \times \mathbb{C}P^{n-1} \to U(n)$ be the map which associates to a pair (λ, ℓ) the rotation of $\mathbb{C}^n$ in the line ℓ with angle of rotation λ. If we let ℓ_0 correspond to the base point of $\mathbb{C}P^{n-1}$, then we can make a map to $SU(n)$ by dividing by the value at ℓ_0: our map is therefore given by $(\lambda, \ell) \mapsto r(\lambda, \ell)r(\lambda, \ell_0)^{-1}$. The points $S^1 \times *$ and $* \times \mathbb{C}P^{n-1}$ all go to the base-point in $SU(n)$, and so this descends to a map

$$S^1 \wedge \mathbb{C}P^{n-1} \to SU(n).$$

Taking the adjoint then gives us a map $\mathbb{C}P^{n-1} \to \Omega SU(n)$ with the property that, when composed with the inclusion $\Omega SU(n) \hookrightarrow \Omega SU \cong BU$, it yields the classifying map for the virtual bundle $L - 1$ over $\mathbb{C}P^{n-1}$ (this is part of an even dimensional cell structure on $\Omega SU(n)$). Finally, applying the Thom spectrum construction to the inclusion $\mathbb{C}P^{n-1} \to \Omega SU(n)$ gives us a map

$$\Sigma^{-2}\mathbb{C}P^{n} \cong (\mathbb{C}P^{n-1})^{L-1} \to X(n). \tag{1}$$

If R is complex orientable, then, by the Thom isomorphism, there is an isomorphism of R_*-algebras

$$R_*X(n) = R_*\Omega SU(n) = \mathrm{Sym}_{R_*}(R_*\mathbb{C}P^{n-1}) = R_*[b_1, \ldots, b_{n-1}].$$

Moreover, multiplicative maps $X(n) \to R$ corresponds bijectively to coordinates up to degree n (that is, modulo degree $n + 1$) on the formal group of R.

Now, even if R is not complex orientable, we can still make sense of multiplicative maps from $X(n)$ to R. They yield "formal group law chunks", that is, formal group laws up to degree n. Assuming the existence of a multiplicative map $X(n) \to R$, we can therefore associate to R a formal group up to degree n.

Recall that a ring spectrum X is called *flat* if X_*X is flat as an X_*-module. Even though the homotopy groups of $X(n)$ are not known for any finite value of n, these spectra are known to be flat, and so there is a sort of Thom isomorphism

$$X(n)_*X(n) = X(n)_*\Omega SU(n) = X(n)_*[b_1, \ldots, b_{n-1}], \qquad |b_i| = 2i.$$

In particular, the left hand side is a free $X(n)_*$-module, and it follows that

$$X(n) \wedge X(n) = X(n)[b_1, \ldots, b_{n-1}],$$

where the last term is a wedge of suspensions of $X(n)$ indexed by the monomials in $\mathbb{Z}[b_1, \ldots, b_{n-1}]$. Like with MU, we can recast this as saying that

$$X(n)_*X(n) = \mathrm{Sym}_{X(n)_*}(X(n)_*\mathbb{C}P^{n-1}).$$

More precisely, $X(n)_*X(n)$ is the symmetric algebra in $X(n)_*$-modules on the $X(n)$-homology of $\Sigma^{-2}\mathbb{C}P^{n}$, where the latter is understood modulo its (-2)-cell. Also, the 0-cell corresponds to $b_0 = 1$, and so doesn't appear among the generators. Note that the map including $X(n)_*\Sigma^{-2}\mathbb{C}P^{n}$ into

$X(n)_*X(n)$ is induced by the map (1) above. As was the case for MU, the pair $\big(X(n)_*, X(n)_*X(n)\big)$ forms a Hopf algebroid. In particular, though we cannot compute the homotopy groups of $X(n)$, we can hope to study algebraically the $X(n)$-based Adams Novikov spectral sequence.

2. Stacks from Spectra

In the old days, in order to talk about formal groups in homotopy theory, one necessarily had to have a complex oriented cohomology theory. However, in retrospect, one can get something having to do with formal groups associated to any spectrum. In this section, given a ring spectrum X, typically not complex orientable, we will associate to it a stack $\mathcal{M}_X$ over $\mathcal{M}_{FG}^{(1)}$.

Here, $\mathcal{M}_{FG}^{(1)}$ denotes the stack classifying formal groups equipped with a first order coordinate or, in other words, a non-zero tangent vector. Recall that the stack associated to the Hopf algebroid (MU_*, MU_*MU), equivalently, to the groupoid $\mathrm{Spec}(MU_*MU) \rightrightarrows \mathrm{Spec}(MU_*)$, is almost but not quite the stack of formal groups. It is $\mathcal{M}_{FG}^{(1)}$. The rings MU_* and MU_*MU being $\mathbb{Z}$-graded, there is an action of $\mathbb{G}_m$ on that stack. The action rescales the tangent vector, and modding out by it yields $\mathcal{M}_{FG}$.

We first consider the case when X is just a spectrum (not a ring spectrum), and use the MU-based Adams resolution

$$MU_*X \rightrightarrows MU_*MU \otimes_{MU_*} MU_*X \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; MU_*MU \otimes_{MU_*} MU_*MU \otimes_{MU_*} MU_*X \;\; \cdots \tag{2}$$

to construct a quasicoherent sheaf $\mathcal{F}_X$ over $\mathcal{M}_{FG}^{(1)}$. We recall how this resolution arises. Start with the cosimplicial spectrum whose k^{th} stage is given by $MU^{\wedge(k+1)} \wedge X$ (inserting the sphere spectrum in various places and then applying the unit map gives the coface maps, and the codegeneracies come from the multiplication on MU), we then apply $\pi_*(-)$ and use repeatedly the isomorphism

$$\pi_*(MU \wedge MU \wedge X) \cong MU_*MU \otimes_{MU_*} MU_*X.$$

The first term of the above resolution is MU_*X, an MU_*-module, and to it corresponds a quasicoherent sheaf over Spec of the Lazard ring $L = MU_*$. Similarly, the next term $MU_*MU \otimes_{MU_*} MU_*X$ gives a sheaf over $\mathrm{Spec}(MU_*MU) = \mathrm{Spec}(L) \times_{\mathcal{M}_{FG}^{(1)}} \mathrm{Spec}(L)$, the moduli scheme that parametrizes a pair of formal group laws, and an isomorphism between the corresponding formal groups that respects the chosen tangent vectors. The two maps

$$MU_*X \rightrightarrows MU_*MU \otimes_{MU_*} MU_*X$$

(the counit and coaction maps) are exactly what one needs to give descent data, and so we get our sheaf $\mathcal{F}_X$ over $\mathcal{M}_{FG}^{(1)}$. Note that everything is $\mathbb{Z}$-graded, which is to say, everything is acted on by $\mathbb{G}_m$, and so one can also get sheaf over $\mathcal{M}_{FG}$ by modding out that action.

If now X is a homotopy commutative ring spectrum, then the terms in the Adams resolution (2) are commutative rings, and $(MU_*X, MU_*MU \otimes_{MU_*} MU_*X)$ is a Hopf algebroid. We define $\mathcal{M}_X$ to be the associated stack. By construction, it comes equipped with a map

$$\mathcal{M}_X \to \mathcal{M}_{FG}^{(1)}.$$

If we want, we could also mod out the action of $\mathbb{G}_m$ to get a stack over $\mathcal{M}_{FG}$. Here are two examples:

- $\mathcal{M}_{S^0} = \mathcal{M}_{FG}^{(1)}$, essentially by definition.
- $\mathcal{M}_{X(n)} = \mathcal{M}_{FG}^{(n)}$, the stack of formal groups together with an n-jet, that is, a coordinate modulo degree $n+1$.

In some sense, the above construction brings all ring spectra into the world of complex orientable cohomology theories, even if they are not themselves complex orientable. If X is complex orientable, then $\mathcal{M}_X$ is actually a scheme: the complex orientation provides a contracting homotopy (in the form of a $(-1)^{\text{st}}$ codegeneracy map), and the Adams-Novikov resolution collapses to π_*X. This means that $\mathcal{M}_X = \mathrm{Spec}(\pi_*X)$, with no stackiness. We can therefore view the stackiness of $\mathcal{M}_X$ as a measure of the failure of complex orientability of X.

If X and Y are ring spectra, then we would like $\mathcal{M}_{X\wedge Y}$ to be the stacky pullback (also called the 2-categorical pullback: a point of the pullback consists of a point of $\mathcal{M}_X$, a point of $\mathcal{M}_Y$, and an isomorphism between their images in $\mathcal{M}_{FG}^{(1)}$)

$$\begin{tikzcd} \mathcal{M}_{X\wedge Y} \arrow[r] \arrow[d] & \mathcal{M}_X \arrow[d] \\ \mathcal{M}_Y \arrow[r] & \mathcal{M}_{FG}^{(1)} \end{tikzcd} \tag{3}$$

as this would allow us to do some key computations. For that, we need to know that $MU_*(X\wedge Y) \cong MU_*(X)\otimes_{MU_*} MU_*(Y)$. If either $MU_*(X)$ or $MU_*(Y)$ is flat as an MU_*-module, then the Künneth spectral sequence for MU collapses, giving us the desired isomorphism. In this case, when we form the resolution for $X\wedge Y$, we see that it is just the tensor product of the resolutions for X and for Y, and this is exactly the desired stacky pullback statement. In short, if MU_*X or MU_*Y is flat as MU_*-module, then (3) is a stacky pullback.

Recall that $\mathcal{M}_X$ is the stack associated to the Hopf algebroid encoded in (2):

$$\mathcal{M}_X = \mathit{Stack}\big(MU_*X, (MU\wedge MU)_*X\big). \tag{4}$$

Here is something that is quite natural from the point of view of stacks: the above presentation of $\mathcal{M}_X$ might be a really inefficient one, and it might be better to replace it with a smaller, more efficient one before attempting any calculations. At the end of the day we care only about the underlying stack (really, just its cohomology), not the particular Hopf algebroid presentation. This gives us lots of flexibility. For example, one doesn't necessarily need to use MU above: if R is a any commutative ring spectrum satisfying the conditions (*i, ii, iii*) below, then we can use R instead of MU in (4), and get an equivalent stack:

$$\mathcal{M}_X = \mathit{Stack}\big(R_*X, (R\wedge R)_*X\big). \tag{5}$$

In the next section, we will apply this trick with $R = X(n)$. This fact, that for any such R, the above resolution gives the same underlying stack, is a reformulation of many of the classical change-of-rings theorems used for computations with the Adams spectral sequence.

Finally, we list the technical conditions that R needs to satisfy for the above story to work:

(*i*) $R\wedge X$ is complex orientable.

(*ii*) R_*R is a flat R_*-module: this guarantees that $(R_*X, (R\wedge R)_*X)$ is a Hopf algebroid.

(*iii*) X is R-local, that is, $R\wedge X \rightrightarrows R\wedge R\wedge X \Rrightarrow \dots$ is a resolution of X. If one prefers a condition that does not depend on X, then one can ask for the sheaf $\mathcal{F}_R$ to have full support (for example, if $\pi_0(R)=\mathbb{Z}$ and R is connective, then this condition is automatically satisfied).

3. Two Important Examples

3.1. The $X(4)$-homology of *tmf*. This computation will tie the Weierstrass Hopf algebroid firmly to *tmf* for topological reasons, and it will allow us to see the usefulness of the stacky pullback square (3). We first recall two things:

- $\mathcal{M}_{X(4)}$ is the moduli stack $\mathcal{M}_{FG}^{(4)}$ of formal groups equipped with a 4-jet.
- $\mathcal{M}_{tmf}$ is the moduli stack $\overline{\mathcal{M}}_{ell}^{+(1)}$ of elliptic curves with a 1-jet, where both multiplicative and additive degenerations allowed.[1]

[1]In our notation, the bar indicates that multiplicative degenerations are allowed, and the plus indicates that additive degenerations are also allowed.

Since $MU_*X(n)$ is flat as MU_*-module, the discussion in Section 2 applies, and so we have a stacky pullback square

$$\begin{tikzcd}
\mathcal{M}_{X(4)\wedge tmf} \arrow[r] \arrow[d] & \mathcal{M}_{tmf} \arrow[d] \\
\mathcal{M}_{X(4)} \arrow[r] & \mathcal{M}_{FG}^{(1)}.
\end{tikzcd}$$

The data classified by $\mathcal{M}_{X(4)\wedge tmf}$ is then an elliptic curve C together with a 4-jet (a coordinate modulo degree 5) on its formal group or, equivalently, a 4-jet on C. This data is exactly what is needed to identify a Weierstrass equation for C. We therefore learn that $\mathcal{M}_{X(4)\wedge tmf}$ is affine:

$$\mathcal{M}_{X(4)\wedge tmf} = \operatorname{Spec}\mathbb{Z}[a_1, a_2, a_3, a_4, a_6],$$

and that

$$\pi_*(X(4) \wedge tmf) = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6].$$

Running a similar analysis for $X(4) \wedge \ldots \wedge X(4) \wedge tmf$, we also learn that the $X(4)$-based Adams resolution of *tmf* is exactly the cobar resolution of the Weierstrass Hopf algebroid.

3.2. **The Stack $\mathcal{M}_{bo}$.** We now investigate the spectrum *bo*, the (-1)-connected cover of KO. This spectrum has been extremely useful in stable homotopy, as computations with it tend to be quite tractable. Its zeroth space is $\mathbb{Z} \times BO$, its 8^{th} space is the 7-connected cover of BO, etc. Its relationship to *bu*, the spectrum of connective complex K-theory, is given by the following result:

Proposition 3.1. *We have an equivalence $\Sigma^{-2}\mathbb{C}P^2 \wedge bo \simeq bu$.*

Proof. Recall that $\Sigma^{-2}\mathbb{C}P^2$ is a cell complex with exactly two cells (a 0-cell and a 2-cell), and attaching map $\eta \in \pi_1(\mathbb{S}^0)$. By the work of Bott, we know that $\Omega(U/O) = \mathbb{Z} \times BO$. The fibration of infinite loop spaces $U/O \to \mathbb{Z} \times BO \to \mathbb{Z} \times BU$ therefore corresponds to a fibration $\Sigma bo \to bo \to bu$ of connective spectra. The first map is multiplication by $\eta \in \pi_1(\mathbb{S}^0) = \pi_1(bo)$, and so

$$bu = \operatorname{cofib}(\eta : \Sigma bo \to bo) = \operatorname{cofib}(\eta : \mathbb{S}^1 \to \mathbb{S}^0) \wedge bo = \Sigma^{-2}\mathbb{C}P^2 \wedge bo.$$

□

Thus, although *bo* is not complex orientable, the spectrum $bo \wedge \Sigma^{-2}\mathbb{C}P^2$ is. It is therefore tempting to try to use $\Sigma^{-2}\mathbb{C}P^2$ for R in equation (5) in order to compute $\mathcal{M}_{bo}$. Moreover, since $\Sigma^{-2}\mathbb{C}P^2$ has exactly 2 cells, this indicates that $\mathcal{M}_{bo}$ should admit a double cover by the scheme $\mathcal{M}_{bu} = \operatorname{Spec}(\pi_* bu)$. Unfortunately, that idea doesn't quite work, because $\Sigma^{-2}\mathbb{C}P^2$ is not a ring spectrum. It is however possible to embed $\Sigma^{-2}\mathbb{C}P^2$ in a nice ring spectrum subject to the conditions (*i, ii, iii*) above, namely $R = X(2)$.

We first explain why $X(2) \wedge bo$ is complex orientable. Recall that by the Thom isomorphism $H_*X(2) = H_*\Omega SU(2) = \mathbb{Z}[b_1]$. It follows that $X(2)$ is a cell complex with one cell in each even degree. In fact, there exists a cell complex Y_4 with a single cell in each degree multiple of 4,

$$H_*Y_4 = \mathbb{Z}[c], \quad |c| = 4,$$

such that $X(2) = Y_4 \wedge \Sigma^{-2}\mathbb{C}P^2$, and therefore

$$X(2) \wedge bo = Y_4 \wedge \Sigma^{-2}\mathbb{C}P^2 \wedge bo = Y_4 \wedge bu$$

is complex orientable.

Corollary 3.2. *As rings*

$$X(2)_*bo = \mathbb{Z}[b, c], \qquad |b| = 2,\ |c| = 4,$$

and

$$X(2)_*X(2) \otimes_{X(2)_*} X(2)_*bo = \mathbb{Z}[b, c, r], \quad |r| = 2.$$

Proof. Since $X(2) \wedge bo = Y_4 \wedge bu$, it follows from the Atiyah Hirzebruch spectral sequence that $X(2)_*bo = bu_*Y_4 = \mathbb{Z}[b, c]$, where $|b| = 2$ and $|c| = 4$. Furthermore, since $X(2)_*X(2) = X(2)_*[b_1]$, base-changing and renaming b_1 to r gives the second part. □

This shows that, while we cannot hope to understand $\pi_* X(2)$, we can understand every piece of the resolution

$$X(2)_* bo \rightrightarrows X(2)_* X(2) \otimes_{X(2)_*} X(2)_* bo \; \substack{\rightarrow\\\rightarrow\\\rightarrow} \dots.$$

Our next goal is to use this to understand the stack

$$\mathcal{M}_{bo} = Stack(X(2)_* bo, (X(2) \wedge X(2))_* bo) = Stack(\mathbb{Z}[b, c], \mathbb{Z}[b, c, r]).$$

Since everything is torsion free, computing the right unit of the Hopf algebroid $\big(X(2)_* bo, X(2)_* X(2) \otimes_{X(2)_*} X(2)_* bo\big)$ turns out not to be too hard: it is given by $b \mapsto b + 2r$ and $c \mapsto c + br + r^2$ (and the left unit is course just $b \mapsto b$, $c \mapsto c$). Those formulas are exactly the transformation rules of the coefficients of $x^2 + bx + c$ under the change of variable $x \mapsto x + r$, and so this Hopf algebroid corepresents the following algebro-geometric objects: curves

$$y = x^2 + bx + c,$$

together with the changes of coordinate $x \mapsto x + r$. Generically, such an equation can be thought of as describing a form of $\mathbb{G}_m$: first projectvize the curve by adding a point at infinity (the neutral element), and then remove the locus where $y = 0$. Moreover, this curve is naturally equipped with an invariant differential $\omega = dx/y$, or equivalently, a 1-jet at infinity. Note the similarity with elliptic curves: if we take the above equation $y^{d-1} = x^d + bx + c$ and set $d = 3$ instead of 2, then we get (generically) an elliptic curve.

We recall what the stack associated to $(\mathbb{Z}[b, c], \mathbb{Z}[b, c, r])$ is. By definition, a stack is given in terms of what it means to map into it. In our case, a map $\operatorname{Spec} A \to \mathcal{M}_{bo}$ consists of 3 pieces of data:

(1) a faithfully flat extension $A \to B$,
(2) a ring homomoprhism $\mathbb{Z}[b, c] \to B$, equivalently, two elements $b, c \in B$, which we'll interpret as giving a curve $y = x^2 + bx + c$ over B,
(3) a ring homomoprhism $\mathbb{Z}[b, c, r] \to B \otimes_A B$ compatible with the left/right actions of $\mathbb{Z}[b, c]$, equivalently, an element $r \in B \otimes_A B$ such that the change of variable $x \mapsto x + r$ yields an isomorphism between the curves over $B \otimes_A B$ gotten by base change along $B \rightrightarrows B \otimes_A B$,

subject to a cocycle condition in $B \otimes_A B \otimes_A B$. In other words, a map $\operatorname{Spec} A \to \mathcal{M}_{bo}$ is descent data for a (form of the) multiplicative group over $\operatorname{Spec} A$, along with a 1-jet. We denote this stack by $\mathcal{M}^{+(1)}_{\mathbb{G}_m}$. It is the moduli stack of multiplicative groups with a 1-jet, where additive degenerations are allowed.

Replacing MU by $X(2)$ in the definition (4) of $\mathcal{M}_{bo}$ was already a huge simplification, and allowed us to identify it with $\mathcal{M}^{+(1)}_{\mathbb{G}_m}$, but there is an even simpler Hopf algebroid that represents this stack. That smaller presentation will also make it evident that the map $\operatorname{Spec} \mathbb{Z}[b] = \mathcal{M}_{bu} \to \mathcal{M}_{bo}$ is finite of degree two, reflecting the fact that $\Sigma^{-2}\mathbb{C}P^2$ has only 2 cells. Consider the map $\operatorname{Spec} \mathbb{Z}[b] \to \mathcal{M}_{bo}$ representing the curve $y = x^2 + bx$. We want to restrict $(\mathbb{Z}[b, c], \mathbb{Z}[b, c, r])$ to curves of only that form, to get a smaller but equivalent Hopf algebroid. To do so, we look at those coordinate transformations $x \mapsto x + r$ that preserve the property $c = 0$, namely those for which r satisfies $r^2 + br = 0$. We therefore get a new Hopf algebroid

$$(\mathbb{Z}[b], \mathbb{Z}[b, r]/r^2 + 2r), \quad b \mapsto b + 2r$$

that represents $\mathcal{M}_{bo}$.

To know that this represents the same stack $\mathcal{M}_{bo}$, we still need to check that $\operatorname{Spec} \mathbb{Z}[b] \to \mathcal{M}_{bo}$ is a flat cover. This can be checked after pulling back along the cover $\operatorname{Spec} \mathbb{Z}[b, c] \to \mathcal{M}_{bo}$ (representing the universal curve $y = x^2 + bx + c$):

$$\begin{tikzcd}
\operatorname{Spec} \mathbb{Z}[b, c, r]/r^2 + br + c \arrow[r] \arrow[d, "\binom{b}{c} \mapsto \binom{b+2r}{0}"'] & \operatorname{Spec} \mathbb{Z}[b] \arrow[d, "y=x^2+bx"] \\
\operatorname{Spec} \mathbb{Z}[b, c] \arrow[r, "y=x^2+bx+c"'] & \mathcal{M}_{bo}
\end{tikzcd}$$

The stacky pullback represents pairs of curves $(y = x^2 + b_1 x, y = x'^2 + bx' + c)$, together with a coordinate transformation $x' = x + r$ that identifies them. This works if and only if the constant coefficient of $(x + r)^2 + b(x + r) + c = x^2 + b_1 x$ is zero, that is, if r is a root of $x^2 + bx + c$. The pullback is therefore given by $\operatorname{Spec} \mathbb{Z}[b, b_1, c, r]/(b+2r-b_1, r^2+br+c) = \operatorname{Spec} \mathbb{Z}[b, c, r]/r^2+br+c$. It is a free module of rank two over $\operatorname{Spec} \mathbb{Z}[b, c]$, and in particular, it is faithfully flat. More generally, adjoining a root of a monic polynomial always produces a faithfully flat extension.

This example with *bo* is an important toy model for understanding how *tmf* works. At $p = 2$, there is an 8-fold cover of the moduli stack $\mathcal{M}_{tmf} = \overline{\mathcal{M}}_{ell}^{+(1)}$ which corresponds to the existence of an 8-cell complex X such that $tmf \wedge X$ is complex orientable (there is also an interesting 24-fold cover). Similarly, at $p = 3$, there is a 3-fold cover of $\mathcal{M}_{tmf}$, corresponding to a 3-cell complex Y such that $tmf \wedge Y$ is complex orientable. These covers and the associated presentations of the moduli stack of elliptic curves are very important for carrying out computations with *tmf*.