$h$-polynomials of triangulations of flow polytopes

Karola Mészáros

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$h$-polynomials of triangulations of flow polytopes

(and of reduction trees)

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Background on flow polytopes

Reduction trees and reduced forms

Reduced forms generalize $h$-polynomials of triangulations

Canonical triangulations of flow polytopes

Shellings and $h$-polynomials of reduction trees

Nonnegativity results on reduced forms

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Flow polytopes

\[ \mathcal{F}_{K_5}(1, 0, 0, 0, -1) \]
Flow polytopes

\[ \mathcal{F}_{K_5}(1,0,0,0,-1) \]

\[
\begin{align*}
1 &= a + b + c + d \\
0 &= e + f + g - a \\
0 &= h + i - b - e \\
0 &= j - c - f - h
\end{align*}
\]

\[ a, b, c, d, e, f, g, h, i, j \geq 0 \]
For a general graph $G$ on the vertex set $[n]$, with net flow $\mathbf{a} = (1, 0, \ldots, 0, -1)$, the flow polytope of $G$, denoted $\mathcal{F}_G$, is the set of flows $f : E(G) \to \mathbb{R}_{\geq 0}$ such that the total flow going in at vertex 1 is one, and there is flow conservation at each of the inner vertices.
Examples of flow polytopes

simplex

$K_4$
An intriguing theorem

**Theorem [Postnikov-Stanley]:**
For a graph $G$ on the vertex set $\{1, 2, \ldots, n\}$ we have

$$\text{vol} \left( \mathcal{F}_G(1, 0, \ldots, 0, -1) \right) = K_G(0, d_2, \ldots, d_{n-1}, -\sum_{i=2}^{n-1} d_i),$$

where $d_i = (\text{indegree of } i) - 1$ and $K_G$ is the Kostant partition function.
Some interesting examples of flow polytopes

**Theorem [Zeilberger 99]:**

$$\text{vol}(F_{K_{n+1}}) = \text{Cat}(1) \cdot \text{Cat}(2) \cdots \text{Cat}(n-2).$$
Some interesting examples of flow polytopes

**Theorem [Zeilberger 99]:**

\[ \text{vol}(\mathcal{F}_{K_{n+1}}) = \text{Cat}(1)\text{Cat}(2) \cdots \text{Cat}(n - 2). \]

\( \mathcal{F}_{K_{n+1}} \) is a member of a larger family of polytopes with volumes given by nice product formulas.
Some interesting examples of flow polytopes

**Theorem [Zeilberger 99]:**

\[ \text{vol}(\mathcal{F}_{K_{n+1}}) = \text{Cat}(1)\text{Cat}(2) \cdots \text{Cat}(n-2). \]

\(\mathcal{F}_{K_{n+1}}\) is a member of a larger family of polytopes with volumes given by nice *product formulas*.

(Think \( \prod_{i=m+1}^{m+n-1} \frac{1}{2i+1} \left(\frac{m+n+i+1}{2i}\right) \).)
Triangulating $\mathcal{F}_G$

$q \geq p$

$p \geq q$

$p = q$
Proposition:

\[ \mathcal{F}_{G_0} = \mathcal{F}_{G_1} \cup \mathcal{F}_{G_2}, \quad \mathcal{F}_{G_1} \cap \mathcal{F}_{G_2} = \mathcal{F}_{G_3}. \]
Triangulating $\mathcal{F}_G$

$q \geq p \quad \quad p \geq q \quad \quad p = q$

$G_0 \quad \quad \quad G_1 \quad \quad \quad G_2 \quad \quad \quad G_3$

Proposition:

$\mathcal{F}_{G_0} = \mathcal{F}_{G_1} \cup \mathcal{F}_{G_2}, \quad \mathcal{F}_{G_1} \cap \mathcal{F}_{G_2} = \mathcal{F}_{G_3}$.

$\mathcal{F}_{G_1}$ or $\mathcal{F}_{G_2}$ could be empty.
\[ \tilde{G} = G \text{ with } s \text{ and } t \]

**Purpose:** we can simply do the reductions on \( G \) and at the end arrive to a triangulation of \( \mathcal{F}_{\tilde{G}} \).
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Back to where we started: last words on flow polytopes
A reduction tree of $G = ([4], \{(1, 2), (2, 3), (3, 4)\})$ with five leaves. The edges on which the reductions are performed are in bold.
Lemma.

If the leaves are labeled by graphs $H_1, \ldots, H_k$ then the flow polytopes $\mathcal{F}_{\overline{H_1}}, \ldots, \mathcal{F}_{\overline{H_k}}$ are simplices.
Lemma. The normalized volume of $\mathcal{F}_{\tilde{G}}$ is equal to the number of leaves in a reduction tree $\mathcal{T}(G)$. 
Reductions in variables

\[ q \geq p \quad p \geq q \quad p = q \]

\[ x_{ij} x_{jk} \rightarrow x_{jk} x_{ik} + x_{ik} x_{ij} + \beta x_{ik} \]
Reduced form

\[ x_{12} x_{23} x_{34} \]
Reduced form

$x_{12}x_{23}x_{34}$
Reduced form

\[ x_{12}x_{23}x_{34} \]

\[ x_{12}x_{13}x_{14} + x_{13}x_{14}x_{24} + x_{13}x_{23}x_{24} + x_{12}x_{14}x_{34} + x_{14}x_{24}x_{34} \]

(\(\beta = 0\))
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Back to where we started: last words on flow polytopes
Denote by $Q_G(\beta, x)$ the reduced form of the monomial
\[ \prod_{(i,j) \in E(G)} x_{ij}. \]
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**Theorem.** (M, 2014)

\[ Q_G(\beta - 1) = h(\mathcal{T}, \beta) \]
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**Theorem.** (M, 2014)

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(where $\mathcal{T}$ is a “triangulation” of $\mathcal{F}_{\tilde{G}}$ obtained via the game)
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**Theorem.** (M, 2014)

$$Q_G(\beta - 1) = h(T, \beta)$$

(\text{where } T \text{ is a “triangulation” of } F_{\tilde{G}} \text{ obtained via the game})

In particular the coefficients of $Q_G(\beta - 1)$ are nonnegative.
Why “triangulation”? 
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If we just play the game in any way we like, we might not get a triangulation in the sense of a simplicial complex.
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Nevertheless, the notions of $f$-vectors and $h$-vectors still make sense.
Why “triangulation”?  

If we just play the game in any way we like, we might not get a triangulation in the sense of a simplicial complex.  

Nevertheless, the notions of $f$-vectors and $h$-vectors still make sense.  

Still, we wonder:  

Is there a way to play the game and get a triangulation?
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Yes, triangulation!

**Theorem.** (M, 2014) There is a way to play the game and obtain a shellable triangulation of the flow polytope $\mathcal{F}_{\tilde{G}}$. 
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A triangulation is said to be **shellable**, if we can order the top dimensional simplices $F_1, \ldots, F_k$, so that $F_i$, $1 < i$, attaches to the preceding simplices $F_1, \ldots, F_{i-1}$, on a union of its facets (at least one of them).
Yes, triangulation!

**Theorem.** (M, 2014) There is a way to play the game and obtain a shellable triangulation of the flow polytope $\mathcal{F}_{\tilde{G}}$.

The key is to use a special reduction order. Namely, do the reductions from left to right and always on the topmost edges.
Yes, triangulation!

**Theorem.** (M, 2014) There is a way to play the game and obtain a shellable triangulation of the flow polytope $\mathcal{F}_{\tilde{G}}$.

The key is to use a special reduction order. Namely, do the reductions from left to right and always on the topmost edges. We call this special order $\mathcal{O}$. 
Reduction tree $R^O_G$
Reduction tree $R_G^O$
Shelling $\mathcal{T}^O$

Let $F_1, \ldots, F_l$ be the full-dimensional leaves of $R_G^O$ ordered by depth-first search order.

**Theorem.** (M, 2014)

$\mathcal{F}_{\tilde{F}_1}, \ldots, \mathcal{F}_{\tilde{F}_l}$ is a shelling order of the triangulation $\mathcal{T}^O$ of $\mathcal{F}_{\tilde{G}}$. 
Let $F_1, \ldots, F_l$ be the full-dimensional leaves of $R_G^O$ ordered by depth-first search order.

**Theorem.** (M, 2014)

$F_{F_1}, \ldots, F_{F_l}$ is a shelling order of the triangulation $T^O$ of $F_\tilde{G}$.

The idea of proof is weak embeddability of reduction trees.
Weak embeddable reduction tree $R^O_G$
Weak embeddable reduction tree $R^O_G$
Weak embeddable reduction tree $R^o_G$
Weak embeddable reduction tree $R^O_G$
Weak embeddable reduction tree $R^O_G$
Weak embeddable reduction tree $R^O_G$
Weak embeddable reduction tree $R_G^O$
Weak embeddable reduction tree $R^O_G$
Leaves of $\mathcal{R}_G^\circ$

\begin{align*}
F_2 \cap F_4 \\
(F_3 \cap F_5) \cap (F_4 \cap F_5)
\end{align*}
Leaves of $R^O_G$

**Theorem.** (M, 2014)

Let $F_1, \ldots, F_l$ be the full-dimensional leaves of $R^O_G$ ordered by depth-first search order. Let

\[
\{Q^i_1, \ldots, Q^i_{f(i)}\} = \{F_i \cap F_j \mid 1 \leq j < i, |E(F_i \cap F_j)| = |E(F_i)| - 1\}.
\]

Then

\[
\sum_{i=1}^l \prod_{j=1}^{f(i)} (F_i + Q^i_j)
\]

is the formal sum of the set of the leaves of $R^O_G$, where the product of graphs is their intersection. If $f(i) = 0$ we define

\[
\prod_{j=1}^{f(i)} (F_i + Q^i_j) = F_i.
\]
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“Shellable” reduction trees

The idea of proof for the previous theorem is to define a notion alike shellability for reduction trees.
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Given a full dimensional leaf $L$ of $R_G$, $H$ is a preceding facet of $L$ if
“Shellable” reduction trees

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Given a full dimensional leaf $L$ of $R_G$, $H$ is a preceding facet of $L$ if

1. $H$ is a leaf before $L$ in $R_G$ in depth-first search order
“Shellable” reduction trees

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Given a full dimensional leaf $L$ of $R_G$, $H$ is a preceding facet of $L$ if

1. $H$ is a leaf before $L$ in $R_G$ in depth-first search order

2. $E(H) \subset E(L)$ and $|E(H)| = |E(L)| - 1$
“Shellable” reduction trees

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1. $H$ is a leaf before $L$ in $R_G$ in depth-first search order

2. $E(H) \subset E(L)$ and $|E(H)| = |E(L)| - 1$

3. *
Strong embeddable reduction tree $R_G^O$
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Strong embeddable reduction tree $R_G^O$
Strong embeddable reduction tree $R^O_G$
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Strong embeddable reduction tree $R^O_G$
$h$-polynomials of reduction trees
 Define the $h$-polynomial of a reduction tree $R_G$ as

$$h(R_G, \beta) = \sum_{i=0}^{\infty} s_i \beta^i,$$

where $s_i$ is the number of full dimensional leaves $L$ of $R_G$ with exactly $i$ preceding facets.
Define the $h$-polynomial of a reduction tree $R_G$ as

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All above can also be defined for partial reduction trees $R^p_G$, or alternatively reduction trees in other algebras.
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All above can also be defined for partial reduction trees $R^p_G$, or alternatively reduction trees in other algebras.

**Theorem.** (M, 2014) For strong embeddable $R^p_G$ we have

$$Q_{R^p_G} (\beta - 1) = h(R^p_G, \beta)$$
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Reduced forms are shifted $h$-polynomials

**Theorem.** (M, 2014) For strong embeddable $R^p_G$ we have

$$Q_{R^p_G}(b - 1) = h(R^p_G, b)$$
Reduced forms are shifted $h$-polynomials

**Theorem.** (M, 2014) For strong embeddable $R_G^p$ we have

$$Q_{R_G^p}(b - 1) = h(R_G^p, b)$$

**Corollary.** (M, 2014) For strong embeddable $R_G^p$ the coefficients of $Q_{R_G^p}(b - 1)$ are nonnegative.
Reduced forms are shifted $h$-polynomials

**Theorem.** (M, 2014) For strong embeddable $R^p_G$ we have

$$Q_{R^p_G}(b - 1) = h(R^p_G, b)$$

**Corollary.** (M, 2014) For strong embeddable $R^p_G$ the coefficients of $Q_{R^p_G}(b - 1)$ are nonnegative.

Generalizations of the above theorem and corollary can be used to address a nonnegativity conjecture of Kirillov.
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If a triangulation is shellable...

Recall that the motivation for the definitions of weak and strong embeddability was the shellable triangulation $T^O$ obtained from $R^O_G$. 
If a triangulation is shellable...

...one wonders if it is regular.
If a triangulation is shellable...

...one wonders if it is regular.

**Question**: Is $\mathcal{T}^O$ regular?
If a triangulation is shellable...

...one wonders if it is regular.

**Question:** Is $\mathcal{T}^O$ regular?

(I am not sure about that, but...)
If a triangulation is shellable...

...one wonders if it is regular.

Question: Is $T^O$ regular?

(I am not sure about that, but... I know something else)
Theorem. (M, 2014) There are ways to play the game and obtain regular and flag triangulations of the flow polytope $\mathcal{F}_{G_{\tilde{G}}}$. 
**Theorem.** (M, 2014) There are ways to play the game and obtain regular and flag triangulations of the flow polytope $\mathcal{F}_{\tilde{G}}$.

This result builds on work of Danilov-Karzanov-Koshevoy.
Happy birthday, Richard!