Cutting polytopes

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Plan of the talk:

1. first example: hypersimplices (slices of the cube):
   - volume,
   - Ehrhart $h$-vector,
   - $f$-vector;

2. second example: edge polytopes;

3. general cutting-polytope framework.
Hypersimplex

The \((k, n)\)th hypersimplex \((0 \leq k < n)\) is

\[
\Delta_{k,n} = \{ \mathbf{x} \in [0, 1]^n \mid k \leq x_1 + \cdots + x_n \leq k + 1 \}.
\]

For example: \(\Delta_{k,3}\)

For any \(n\)-dimensional polytope \(\mathcal{P}\), its normalized volume: \(\text{nvol}(\mathcal{P}) = n! \text{vol}(\mathcal{P})\). E.g., the unit cube \(C = [0, 1]^n\) has \(\text{nvol}(C) = n!\).
Normalized volume of $\Delta_{k,n}$

**Theorem (Laplace)**

\[ \text{nvol} \Delta_{k,n} = \# \{ w \in S_n \mid \text{des}(w) = k \} , \text{ which provides a refinement of } \text{nvol}([0, 1]^n). \]

Stanley gave a bijective proof in 1977 (the shortest paper).

**Example**

\[ \text{nvol}(\Delta_{1,3}) = 4, \text{ and } S_3 = \{ 123, 213, 312, 132, 231, 321 \}. \]
**Ehrhart $h$-vector**

$\mathcal{P} \subset \mathbb{R}^N$: an $n$-dimensional integral polytope. E.g., for the unit square, we have $\#(r\mathcal{P} \cap \mathbb{Z}^2) = (r + 1)^2$, for $r \in \mathbb{P}$.

- **Ehrhart polynomial**: $i(\mathcal{P}, r) = \#(r\mathcal{P} \cap \mathbb{Z}^N)$.

  $$
  \sum_{r \geq 0} i(\mathcal{P}, r)t^r = \frac{h(t)}{(1 - t)^{n+1}}.
  $$

- **$h$-polynomial**: $h(t) = h_0 + h_1 t + \cdots + h_n t^n$

- **$h$-vector**: $(h_0, \ldots, h_n)$. $h_i \in \mathbb{Z}_{\geq 0}$ (Stanley).

  $$
  \sum_{i=0}^{n} h_i = n\text{vol}(\mathcal{P}).
  $$
Ehrhart $h$-vector

Ehrhart $h$-vector of $\mathcal{P}$ provides a refinement of its normalized volume. For example,

- for the unit cube $[0, 1]^n, h_i = \# \{ w \in S_n \mid \text{des}(w) = i \};$
- for the hypersimplex $\text{nvol} \Delta_{k,n} = \# \{ w \in S_n \mid \text{des}(w) = k \}.$

$h_i = \text{?}$

**Key point** (Stanley): study the half-open hypersimplex instead of the hypersimplex.

**Definition**

The **half-open hypersimplex** $\Delta'_{k,n}$ is defined as: $\Delta'_{1,n} = \Delta_{1,n}$ and if $k > 1,$

$$\Delta'_{k,n} = \{ x \in [0, 1]^n \mid k < x_1 + \cdots + x_n \leq k + 1 \}.$$
Ehrhart $h$-vector of the half-open hypersimplex

Let $\text{exc}(w) = \#\{i \mid w(i) > i\}$, for any $w \in \mathcal{S}_n$. For $\Delta'_{k,n}$,

**Theorem (L. 2012, conjectured by Stanley)**

$h_i = \#\{w \in \mathcal{S}_n \mid \text{exc}(w) = k \text{ and } \text{des}(w) = i\}$.

**Example**

<table>
<thead>
<tr>
<th>$w$</th>
<th>123</th>
<th>132</th>
<th>213</th>
<th>231</th>
<th>312</th>
<th>321</th>
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<td>1</td>
<td>2</td>
</tr>
<tr>
<td>exc</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

- for $\Delta'_{0,3}$, $k = 0$, $h(t) = 1$;
- for $\Delta'_{1,3}$, $k = 1$, $h(t) = 3t + t^2$;
- for $\Delta'_{2,3}$, $k = 2$, $h(t) = t$. 
Ehrhart \( h \)-vector of the half-open hypersimplex

Equivalently, the \( h \)-polynomial of \( \Delta'_{k,n} \) is

\[
\sum_{w \in \mathcal{S}_n \text{ exc}(w)=k} t^{\text{des}(w)}.
\]

Two proofs:

- generating functions, based on a result by Foata and Han;
- by a unimodular shellable triangulation, and

**Theorem (Stanley, 1980)**

Assume an integral \( \mathcal{P} \) has a shellable unimodular triangulation \( \Gamma \).

For each simplex \( \alpha \in \Gamma \), let \( \#(\alpha) \) be its shelling number. Then \( h \)-polynomial of \( \mathcal{P} \) is

\[
\sum_{\alpha \in \Gamma} t^{\#(\alpha)}.
\]
**$f$-vector of the half-open hypersimplex**

Let $f'_j(n,k)$ denote the number of $j$-faces of $\Delta'_{n,k}$.

**Property (Hibi, L. and Ohsugi, 2013)**

*The sum of $f$-vectors for the half-open hypersimplex (also the $f$-vector of the hypersimplical decomposition of the unit cube) is*

$$
\sum_{k=0}^{n-1} f'_j(n,k) = j \cdot 2^{n-j-1} \frac{n+j+2}{n+1} \cdot \binom{n+1}{j+1}.
$$

**Question**

*Connection with Chebyshev polynomials?*

Fix $j = 2$, $\frac{1}{j} \sum_{k=0}^{n-1} f'_j(n,k) = 1, 7, 32, 120, 400, 1232, 3584, \ldots$, appears in the triangle table of coefficients of Chebyshev polynomials of the first kind (by OEIS).
For a polytope $\mathcal{P}$ (assume convex and integral),

1. **decomposability** can we cut it into two integral subpolytopes with the same dimension by a hyperplane (called separating hyperplane);

2. **inheritance** do the subpolytopes have the same nice properties as $\mathcal{P}$?

3. **equivalence** can we count or classify all the different decompositions?
Cutting edge polytopes

Definition
Let $G$ be a connected finite graph with $n$ vertices and edge set $E(G)$. Then define the edge polytope for $G$ to be

$$P_G = \text{conv}\{e_i + e_j \mid (i, j) \in E(G)\}.$$ 

Combinatorial and algebraic properties of $P_G$ are studied by Ohsugi and Hibi. Based on their results, we study the following question.

Question
Is $P_G$ decomposable or not; can we classify all the separating hyperplanes?
Decomposible edge polytopes

Property (Hibi, L. and Zhang, 2013)

Any separating hyperplanes of edge polytopes have one the following two forms: \( a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0 \), with \( a_i \in \{-1, 0, 1\} \), and for each pair of edge \((i, j)\), \((a_i, a_j)\) either

1. type I: \((1, 1), (-1, 1)\) or \((-1, -1)\);
2. or type II: \((1, 0), (0, 0)\) or \((-1, 0)\).

Property (Funato, L. and Shikama, 2014)

- Infinitely many graphs in each case: 1) type I not II, 2) type II not I, 3) both type I and II, 4) neither type I nor II.
- For bipartite graphs \( G \), type I and II are equivalent.
Decomposable edge polytopes

If $P_G$ is decomposable via a separating hyperplane $H$, then

- $P_G = P_{G_+} \cup P_{G_-}$ where $G = G_+ \cup G_-;$
- $P_G \cap H = P_{G_+} \cap P_{G_-} = P_{G_0}$ where $G_0 = G_+ \cap G_-.$

Property (Funato, L. and Shikama, 2014)

*Characterization of decomposable $G$ in terms of $G_0$:*

- if $G$ bipartite (both type I and type II), then $G_0$ has two connected components, both bipartite;

- if $G$ not bipartite, then
  1. if $G$ is type I, then $G_0$ is a connected bipartite graph;
  2. if $G$ is type II, then $G_0$ has two connected components, one bipartite, the other not.
Normal edge polytopes

Definition
We call an integral polytope $P \subset \mathbb{R}^d$ normal if, for all positive integers $N$ and for all $\beta \in NP \cap \mathbb{Z}^d$, there exist $\beta_1, \ldots, \beta_N$ belonging to $P \cap \mathbb{Z}^d$ such that $\beta = \sum_i \beta_i$.

Theorem (Hibi, L. and Zhang, 2013)
If $P_G$ can be decomposed into $P_{G+} \cup P_{G-}$, then $P_G$ is normal if and only if both $P_{G+}$ and $P_{G-}$ are normal.
General framework

Let \( P \) be a convex and integral polytope and not a simplex.

1. Can we cut it into two integral subpolytopes? E.g.,
   - edge polytopes;
   - *order polytopes, chain polytopes (Yes);
   - *Birkhoff polytopes (No).

2. Do the subpolytopes have the same nice properties as \( P \)?
   - Algebraic properties: normality, quadratic generation of toric ideals;
   - combinatorial properties: volume, \( f \)-vector, \( h \)-vector.

3. Can we count or classify all the decompositions? E.g.,
   - *cutting cubes by two hyperplanes;
   - *order polytopes and chain polytopes for some special posets.

* In a recent work with Hibi.