Nabla and Tors

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The Haiman/Bridgeland-King-Reid map

\[ R\Gamma(P \otimes -) : \{(\text{complexes of}) \text{ sheaves } E \text{ on } \text{Hilb}_n \mathbb{C}^2\} \leftrightarrow \{(\text{complexes of}) \mathbb{C}[x, y] \rtimes S_n\text{-modules } M\} \]

\[ x, y = \{x_1, \ldots, x_n\}, \{y_1, \ldots, y_n\} \]

isomorphism at the level of derived categories. Interested in Tor-groups \( \text{Tor}^i_{\mathbb{C}[x, y]}(M, \mathbb{C}) \) as bigraded \( S_n\)-representations. For instance,

\[ \text{Tor}^0_{\mathbb{C}[x, y]} \Gamma(P \otimes P) = DR_n = \mathbb{C}[x, y]/\langle \sum_{i=1}^n x_i^r y_i^s : (r, s) \neq (0, 0) \rangle \]

This talk: formula for the character of the equivariant index

\[ \chi = \sum_i (-1)^i \text{Tor}_i, \text{ second formula has to do with resolution.} \]
### Examples

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Hilbert scheme

$$\text{Hilb}_n = \{ I \subset \mathbb{C}[x, y] : \dim(R/I) = n \},$$
smooth, dimension $2n$. There is an open subset

$$U_n = \left\{ (p_1 \neq \cdots \neq p_n) \subset (\mathbb{C}^2)^n \right\} / S_n \subset \text{Hilb}_n$$

and the punctual Hilbert scheme

$$Z_n = \pi^{-1}(0), \quad \pi : \text{Hilb}_n \to (\mathbb{C}^2)^n / S_n$$

which is $\mathbb{CP}^1$ for $n = 2$, singular for $n > 2$. The standard torus action

$$T = \mathbb{C}^* \times \mathbb{C}^* \circ \text{Hilb}_n$$
is induced from

$$(q, t) \cdot (x, y) = (q^{-1}x, t^{-1}y)$$
There is a bundle $P$ on $\text{Hilb}_n$ whose restriction to the open subset $U_n \subset \text{Hilb}_n$ by

$$P|_{U_n} = \{(p_1 \neq \cdots \neq p_n)\} \times s_n \mathbb{C}[S_n]$$

- $P$ can be extended to a vector bundle on all of $\text{Hilb}_n$.
- $S_n \curvearrowright P$, fibers $\cong$ regular representation.
- $P^{S_{n-1} \times S_1} = B$ where
  $$\text{rank}(B) = n, \quad B|_I = \mathbb{C}[x, y]/I$$
- $P^{\text{sign}} = \mathcal{O}(1) = \det(B)$
- Procesi bundles are more general objects in the categorical McKay correspondence (see Loseu).
**Garsia-Haiman module**

Torus-fixed points of Hilb$_n$ are monomial ideals, spanned by all $x^iy^j$ for $i, j$ outside a Young diagram $\mu$. Fibers of $P$ over torus-fixed points $\mu$ are the Garsia-Haiman modules: Given $\mu$, order the boxes $(a_1, b_1)\ldots(a_n, b_n)$ in some way, i.e.

$$\mu = [3, 3, 1] \rightarrow \{(a_i, b_i)\} = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0)\}$$

Then we set

$$A_\mu = (a_{ij}), \quad a_{i,j} = x_i^{a_j}y_i^{b_j}$$

$$P_\mu = \langle f(\partial x_i, \partial y_j) \det(A_\mu) \rangle$$

If $\mu = [2]$ then

$$A_{[2]} = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix}, \quad P_\mu = \langle 1, x_1 - x_2 \rangle$$
Modified Macdonald polynomials

The Frobenius character of $P_\mu$ is the modified Macdonald polynomial (part of Haiman’s proof of $n!$ conjecture):

$$H_\mu = \mathcal{F}P_\mu = \sum_\lambda m_\lambda \, \text{ch}_{q,t} \ P_\mu^{S_{\lambda_1} \times \cdots \times S_{\lambda_l}} =$$

$$\sum_\lambda s_\lambda \, \text{ch}_{q,t} \langle P_\mu, \chi^\lambda \rangle$$

Here $P_{\mu}^{S_\lambda}$ is the invariant subspace of the Young subgroup, and $\langle P_\mu, \chi^\lambda \rangle$ is the multiplicity of the irreducible representation $\chi^\lambda$ of $S_n$. For instance,

$$H_{[2]} = m_{[2]} + (1 + q)m_{[1,1]} = s_{[2]} + qs_{[1,1]}$$

so $P_{[2]}$ has one component of the trivial representation in degree $(0,0)$, one component of the sign rep in degree $(1,0)$. 
Localization

The equivariant Euler characteristic of \( E = P^* \otimes P^{l-1} \otimes \mathcal{O}(k) \) is given by

\[
\chi_{\text{Hilb}_n}(E) := \sum_{i \geq 0} (-1)^i \mathcal{F} H^i(E) = \sum_{|\mu| = n} \frac{H_\mu[X^1] \cdots H_\mu[X^l](q^{n(\mu)} t^{n(\mu')})^k}{(H_\mu, H_\mu)^*} \]

\[
\nabla^k \chi_i \sum_{|\mu| = n} \frac{H_\mu[X^1] \cdots H_\mu[X^l]}{(H_\mu, H_\mu)^*},
\]

where

\[
\nabla H_\mu = q^{n(\mu)} t^{n(\mu')} H_\mu.
\]

Conjecture (Haiman, Bergeron, Garsia, Tesler)

\[ \langle \nabla^k s_\mu, s_\nu \rangle \text{ is signed-positive} \]
For $l = 2$, the above sum is

$$\chi(P^* \otimes P \otimes \mathcal{O}(k)) = \nabla^k h_n \left[ \frac{XY}{(1 - q)(1 - t)} \right]$$

where

$$f \left[ \frac{XY}{(1 - q)(1 - t)} \right] = f\bigg|_{p_k = p_k(X)p_k(Y)/(1-q^k)(1-t^k)}$$

If $f(X) = \mathcal{F}(M)$ for a $\mathbb{C}[x] \rtimes S_n$ module-$M$, then

$$f[(1 - q)X] = \sum_i (-1)^i \mathcal{F} \text{Tor}_i^{\mathbb{C}[x]} M$$

Haiman gives $J =$ space of sections as an explicit $S_n \times S_n$-equivariant module over $4n$ variables $x, y, z, w$. For $k = 1$

$$J = \mathbb{C}[x, y, z, w]/ \bigcap_{\sigma \in S_n} \ker_{\mathbb{C}[x, y, z, w]}(f \mapsto f(x, y, z_\sigma, w_\sigma))$$
Fix $N$, and let

$$(m, a, b) \in \mathbb{Z}_{\geq 0}^n \times \{1, \ldots, N\}^n \times \{1, \ldots, N\}^n$$

be sorted so that $m_i \geq m_j$, $m_i = m_j \Rightarrow a_i \leq a_j$, $m_i = m_j, a_i = a_j \Rightarrow b_i \leq b_j$.

\[
dinv_k(m, a, b) = \sum_{i<j} \max(k - 1 + m_j - m_i + \delta(a_i > a_j) + \delta(b_i > b_j), 0)
\]
Theorem (Mellit, C)

\[ \nabla^k_X h_n \left[ \frac{\pm XY}{(1 - q)(1 - t)} \right] = \sum_{[m,a,b]} t^{|m|} q^{dinv_k(m,a,b^\pm)} h_{\mu(m,a,b)} \left[ \frac{1}{1 - q} \right] X_a Y_b \]

Theorem (M,C)

\[ \nabla^k_X h_n \left[ \frac{-XY}{1 - q} \right] = \sum_{[m,a,S \subseteq \{1, \ldots, n\}, b]} (-1)^{|S|} above \text{ formula, but } b_i \text{ is "odd" if } i \in S \]
Affine Springer fiber

Let

\[ \mathcal{F}l_n = \{ \mathbb{C}^n((t)) \supset \cdots \supset \Lambda_i \supset \Lambda_{i+1} \supset \cdots : \]

\[ \dim(\Lambda_i/\Lambda_{i+1}) = 1, \ \Lambda_{i+n} = t\Lambda_i, \ \text{ind}(\Lambda_0) = 0 \} = SL_n((t))/I \]

\[ X_\gamma = \{ g \cdot I \in \mathcal{F}l_n : g^{-1}\gamma g \in \text{Lie}(I) \} , \]

where in our case, \( \gamma = \text{diag}(a_1 t^k, \ldots, a_n t^k) \).

- The torus action by \( T \subset SL_n \subset SL((t)) \) on \( \mathcal{F}l_n \) preserves \( X_\gamma \),
  and the fixed set is the entire affine Weyl group \( W \).

- The equivariant homology is characterized by relations in \( i_*^{-1}(H_*(X_\gamma)) \subset \mathbb{C}(x_1, \ldots, x_n) \cdot W \) by GKM.

- There is a left and right action of \( W, \mathbb{C}[x] \), and \( \mathbb{C}[z] \) on homology.

- Homology is free over \( \mathbb{C}[x] \), there is an affine paving, Schubert type basis.
**Lattice quotient**

The lattice $\mathbb{Z}^n \subset W$ acts space-level on $X_\gamma$. GKM study the quotient $H_*(\mathbb{Z}^n \backslash X_\gamma)$. No longer has affine paving but is compact. The $n = 2$ case is here:
GKM construction

Theorem (GKM)

\[ H_m(\mathbb{Z}^n \setminus X_\gamma) = \bigoplus_{p+q=m} \text{Tor}^{\mathbb{Z}^n}_p (H_q(X_\gamma)) \]

Proof.
The Cartan-Leray spectral sequence collapses. \(\square\)
Relation with $J$

We have

$$J = \text{Im} \left( \pi : \mathbb{C}[x, y, z, w] \to \bigoplus_{\sigma \in S_n} \mathbb{C}[x, y] \sigma \right)$$

$$\pi(f) = \sum_{\sigma} f(x, y, z_\sigma, w_\sigma) \sigma$$

since the kernel is the intersection. There’s a map

$$J[y^{-1}] \to \bigoplus_{w \in W} \mathbb{C}(x)w, \quad f(x)y^a \sigma \mapsto \frac{f(x)}{\prod_{i < j}(x_i - x_j)}(a \cdot \sigma).$$

Claim: the image is precisely $H^T_*(X_\gamma)$. Oscar Kivinen showed that the GKM relations are satisfied by $J$ for the case of the sign rep (Grassmannian case), showing $J \subset H^T_*(X_\gamma)$. Other inclusion is difficult, essentially follows from Haiman’s papers.
Main points:

- $\text{dinv}_k(m, a)$ is the dimension of the cell corresponding to $w(m, a)$.
- $J$ is free over both $\mathbb{C}[x]$ and $\mathbb{C}[y]$, but not $\mathbb{C}[x, y]$. Freeness over $x$ corresponds to equivariant formality.
- Tensoring out over $x$ with $\mathbb{C}$ is like passing from equivariant to non-equivariant homology.
- Tensoring over $\mathbb{C}[y]$ with $\mathbb{C}$ is more subtle. In the GKM formula, you have that $y \in \mathbb{Z}^n$ acts by 1 on $\mathbb{C}$, not 0. This means you lose the $t$ grading, which does not exist in GKM Theorem.
Replace $H_\ast(X_\gamma)$ on the right hand side with $J$, and $\mathbb{Z}^n$ with $\mathbb{C}[y]$ on the right-hand side.

$$\sum_{i} x^i \text{Tor}_i^{C[x,y]} J_2 = a_2 b_2 + (q + t) a_2 b_{1,1} + tqa_{1,1}b_{1,1} x \to 1 + x + 2x^2$$

at $f \mapsto x^\dim(X_\gamma)(f|_{t=1, q=1/x^2})$, (forgetting $S_n$-action).

Observation: this agrees with the Betti numbers of two spheres glued together.
$n = 3$ case

$$\sum_i x^i \Tor_i^{C[x,y]} J_3 = (q^3 a_3 b_3 + q^2 t a_3 b_3 +$$

$$q t^2 a_{1,1,1} b_3 + t^3 a_3 b_3 + q^2 a_3 b_{2,1} +$$

$$q t a_{1,1,1} b_3 + q t a_3 b_{2,1} +$$

$$t^2 a_3 b_{2,1} + q a_{1,1,1} b_{2,1} + t a_3 b_{2,1} + a_3 b_{1,1,1}) +$$

$$(q^3 t a_{2,1} b_3 + q^2 t^2 a_{2,1} b_3 + q t^3 a_{2,1} b_3 +$$

$$q^2 t a_{2,1} b_{2,1} + q t^2 a_{2,1} b_{2,1}) x +$$

$$(q^3 t^2 a_{1,1,1} b_3 + q^2 t^3 a_{1,1,1} b_3 + q^2 t^2 a_{1,1,1} b_{2,1}) x^2 \rightarrow$$

$$6 x^6 + 6 x^5 + 9 x^4 + 6 x^3 + 4 x^2 + 2 x + 1$$
Proofs of formulas

- First formula: current proof, modified vector bundle counting method discovered earlier by Mellit.
- Once formula is discovered, should be many proofs. Potential second proof, first formula can be taken as definition. Other conjectures show that it satisfies defining properties.
- Proof of second formula: first formula.
- Idea behind second formula: tensor over $\mathbb{C}[y]$ first.
- Like taking equivariant homology of $\mathbb{Z}^n \setminus X_\gamma$. 
Garsia-Stanton descent order

Method for getting “dinv” type formulas. Suppose you have a module $J$ over $\mathbb{C}[x, y] \rtimes S_n$. Define a filtration on $J$ by

$$F_a J = \langle y^b : b \leq_{\text{des}} a \rangle_{\mathbb{C}[x]}$$

where for $a, b \in \mathbb{Z}^n_{\geq 0}$, we have $a \leq_{\text{des}} b$ if

1. $\text{sort}(a, >) <_{\text{lex}} \text{sort}(b, >)$
2. $\text{sort}(a, >) = \text{sort}(b, >)$ and $a \leq_{\text{lex}} b$.

Conjecture

In many situations, $\text{ch}_{q,t} F_a J/F_{a-1} J$ has a combinatorial dinv type formula.

Can be used to predict combinatorial formulas, but might be harder to prove the conjecture than the formula.
Theorem (C, Oblomkov)

Let $J = DR_n$, $F_a = F_a J$. Then either $F_a/F_{a-1}$ is nonzero for only $n!$ different choices of $a$, namely

$$a = a(\tau), \quad y^{a(\tau)} := \prod_{\tau_i > \tau_{i+1}} y_{\tau_1} \cdots y_{\tau_i}.$$ 

In this case

$$\text{ch}_{q,t} F_a/F_{a-1} = \prod_i [w_i(\tau)]_q$$

where $w_i(\tau)$ is the number of $\tau_j$ greater than $\tau_i$ in the “run” containing $\tau_i$, plus the number of $\tau_j$ less than $\tau_i$ in the next run. Moreover, $F_a/F_{a-1}$ has an explicit description as modules over $\mathbb{C}[x]$ in terms of the homology of certain Hessenberg varieties.
Proof.
Relate the subquotients to a different affine Springer fiber, descent order to Bruhat order. The combinatorial description was known to several authors, including Gorsky, Mazin, Hikita,...Missing part of the argument was to define the $t$ grading.
Example: $w_4(2, 9, 6, 4, 5, 8, 1, 3, 7) = 2 + 2 = 4$. We have
\[
ch_{q,t} \, DR_3 = (1+q)(1+q+q^2)+t^2+t(1+q)+t^2(1+q)+t(1+q)^2+t^3
\]
Example: Hall-Littlewood polynomials

**Theorem (C)**

Let $J = \Gamma_{\text{Hilb}_n} \mathbb{S}_\lambda$. Then

$$
\text{ch}_{q,t} J = \sum_a t^{|a|} \sum_{\mu} q^* \langle HL_a, s_\lambda HL_\mu \rangle
$$

The inner product is the matrix element of multiplication by $s_\lambda$ in the Hall-Littlewood basis. It’s positive valued.

**Conjecture**

$t^{|a|} \text{ch}_q F_a J / F_{a-1}$ is the summand
Conjecture

The summand in the first theorem is $t^{|m|} \chi_q F_m J / F_{m-1} J$ where

$$F_m J = \langle y^{\sigma(m)} \sigma \rangle.$$ 

Conjecture

The summand in the second theorem is $t^{|m|} \chi_q F_m y^S J / F_{m-1} y^S J$ where

Conjecture

The summand in the second formula

$$\chi_k(m, a) = \sum_{S, b} (-1)^{|S|} q^{\text{dinv}_k(m, a, \tilde{b})} h_{\mu(m, a, b)} \left[ \frac{1}{1 - q} \right] X_a Y_b$$

is Schur-positive. The substitution $\chi_k(m, a)[Y(1 - q)]$ is signed-positive.
Happy birthday Andrei and Pavel!