Long-term dynamics of nonlinear wave equations

W. Schlag (University of Chicago)

Recent Developments & Future Directions, September 2014
Wave maps

Let \((M, g)\) be a Riemannian manifold, and \(u : \mathbb{R}^{1+d}_{t,x} \to M\) smooth.

Wave maps defined by Lagrangian

\[
\mathcal{L}(u, \partial_t u) = \int_{\mathbb{R}^{1+d}_{t,x}} \frac{1}{2} (-|\partial_t u|^2_g + \sum_{j=1}^d |\partial_j u|^2_g) \, dt \, dx
\]

Critical points \(\mathcal{L}'(u, \partial_t u) = 0\) satisfy “manifold-valued wave equation”. \(M \subset \mathbb{R}^N\) embedded, this equation is

\[
\Box u \perp T_u M \text{ or } \Box u = A(u)(\partial u, \partial u),
\]

\(A\) being the second fundamental form.

For example, \(M = S^{n-1}\), then

\[
\Box u = u(|\partial_t u|^2 - |\nabla u|^2)
\]

Note: Nonlinear wave equation, null-form! Harmonic maps are solutions.
Wave maps

Intrinsic formulation: \( D^\alpha \partial_\alpha u = \eta^{\alpha\beta} D_\beta \partial_\alpha u = 0 \), in coordinates

\[ -\partial_{tt} u^i + \Delta u^i + \Gamma^i_{jk}(u) \partial_\alpha u^j \partial^\alpha u^k = 0 \]

\( \eta = (-1, 1, 1, \ldots, 1) \) Minkowski metric

- Similarity with geodesic equation: \( u = \gamma \circ \varphi \) is a wave map provided \( \Box \varphi = 0 \), \( \gamma \) a geodesic.

- Energy conservation: \( E(u, \partial_t u) = \int_{\mathbb{R}^d} \left( |\partial_t u|^2_g + \sum_{j=1}^d |\partial_j u|^2_g \right) dx \) is conserved in time.

- Cauchy problem:

\[ \Box u = A(u)(\partial^\alpha u, \partial_\alpha u), \quad (u(0), \partial_t u(0)) = (u_0, u_1) \]

smooth data. Does there exist a smooth local or global-in-time solution?

Local: Yes. Global: depends on the dimension of Minkowski space and the geometry of the target.
Criticality and dimension

If $u(t, x)$ is a wave map, then so is $u(\lambda t, \lambda x)$, $\forall \lambda > 0$.

Data in the Sobolev space $\dot{H}^s \times \dot{H}^{s-1}(\mathbb{R}^d)$. For which $s$ is this space invariant under the natural scaling? Answer: $s = \frac{d}{2}$.

Scaling of the energy: $u(t, x) \mapsto \lambda^{\frac{d-2}{2}} u(\lambda t, \lambda x)$ same as $\dot{H}^1 \times L^2$.

- **Subcritical case:** $d = 1$. The natural scaling is associated with less regularity than that of the conserved energy. Expect global existence. Logic: local time of existence only depends on energy of data, which is preserved.

- **Critical case:** $d = 2$. Energy keeps the balance with the natural scaling of the equation. For $\mathbb{S}^2$ can have finite-time blowup, whereas for $\mathbb{H}^2$ have global existence. Krieger-S.-Tataru 06, Krieger-S. 09, Rodnianski-Raphael 09, Sterbenz-Tataru 09.

- **Supercritical case:** $d \geq 3$. Poorly understood. Self-similar blowup $Q(r/t)$ for sphere as target, Shatah 80s. Also negatively curved manifolds possible in high dimensions: Cazenave, Shatah, Tahvildar-Zadeh 98.
A nonlinear defocusing Klein-Gordon equation

Consider in $\mathbb{R}_{t,x}^{1+3}$

$$\Box u + u + u^3 = 0, \quad (u(0), \dot{u}(0)) = (f, g) \in \mathcal{H} := H^1 \times L^2(\mathbb{R}^3)$$

Conserved energy

$$E(u, \dot{u}) = \int_{\mathbb{R}^3} \left( \frac{1}{2} |\dot{u}|^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u|^2 + \frac{1}{4} |u|^4 \right) dx$$

With $S(t)$ the linear propagator of $\Box + 1$ we have

$$\ddot{u}(t) = (u, \dot{u})(t) = S(t)(f, g) - \int_0^t S(t-s)(0, u^3(s)) \, ds$$

whence by a simple energy estimate, $I = (0, T)$

$$\|\ddot{u}\|_{L^\infty(I;\mathcal{H})} \leq \|(f, g)\|_{\mathcal{H}} + \|u^3\|_{L^1(I;L^2)} \leq \|(f, g)\|_{\mathcal{H}} + \|u\|_{L^3(I;L^6)}^3$$

$$\leq \|(f, g)\|_{\mathcal{H}} + T\|\ddot{u}\|_{L^\infty(I;\mathcal{H})}^3$$

Contraction for small $T$ implies local wellposedness for $\mathcal{H}$ data.
Defocusing NLKG3

$T$ depends only on $\mathcal{H}$-size of data. From energy conservation we obtain global existence by time-stepping.

Scattering (as in linear theory): $\|\vec{u}(t) - \vec{v}(t)\|_{\mathcal{H}} \to 0$ as $t \to \infty$ where $\Box v + v = 0$ energy solution.

$$\vec{v}(0) := \vec{u}(0) - \int_0^\infty S(-s)(0, u^3)(s) \, ds \text{ provided } \|u^3\|_{L^1_t L^2_x} < \infty$$

Strichartz estimate uniformly in intervals $I$

$$\|\vec{u}\|_{L^\infty(I; \mathcal{H})} + \|u\|_{L^3(I; L^6)} \lesssim \|(f, g)\|_{\mathcal{H}} + \|u^3\|_{L^3(I; L^6)}^3$$

Small data scattering: $\|\vec{u}\|_{L^3(I; L^6)} \lesssim \|(f, g)\|_{\mathcal{H}} \ll 1$ for all $I$. So $I = \mathbb{R}$ as desired.

Large data scattering valid; induction on energy, concentration compactness (Bourgain, Bahouri-Gerard, Kenig-Merle).
Scattering blueprint

Let \( \tilde{u} \) be nonlinear solution with data \( (u_0, u_1) \in \mathcal{H} \). Forward scattering set

\[
S_+ = \{(u_0, u_1) \in \mathcal{H} \mid \tilde{u}(t) \text{ exists globally, scatters as } t \to +\infty\}
\]

We claim that \( S_+ = \mathcal{H} \). This is proved via the following outline:

- **(Small data result):** \( \|(u_0, u_1)\|_{\mathcal{H}} < \varepsilon \) implies \( (u_0, u_1) \in S_+ \)

- **(Concentration Compactness):** If scattering fails, i.e., if \( S_+ \neq \mathcal{H} \), then construct \( \tilde{u}_* \) of minimal energy \( E_* > 0 \) for which \( \|u_*\|_{L_t^3 L_x^6} = \infty \). There exists \( x(t) \) so that the trajectory

\[
K_+ = \{\tilde{u}_*(\cdot - x(t), t) \mid t \geq 0\}
\]

is pre-compact in \( \mathcal{H} \).

- **(Rigidity Argument):** If a forward global evolution \( \tilde{u} \) has the property that \( K_+ \) pre-compact in \( \mathcal{H} \), then \( u \equiv 0 \).

Let \( \{u_n\}_{n=1}^{\infty} \) free Klein-Gordon solutions in \( \mathbb{R}^3 \) s.t.

\[
\sup_n \|\tilde{u}_n\|_{L_t^\infty L_x^H} < \infty
\]

\( \exists \) free solutions \( v^j \) bounded in \( \mathcal{H} \), and \( (t^j_n, x^j_n) \in \mathbb{R} \times \mathbb{R}^3 \) s.t.

\[
u_n(t, x) = \sum_{1 \leq j < J} v^j(t + t^j_n, x + x^j_n) + w^J_n(t, x)
\]

satisfies \( \forall j < J, \tilde{w}^J_n(-t^j_n, -x^j_n) \to 0 \) in \( \mathcal{H} \) as \( n \to \infty \), and

- \( \lim_{n \to \infty} (|t^j_n - t^K_n| + |x^j_n - x^K_n|) = \infty \) \( \forall j \neq k \)
- dispersive errors \( w^k_n \) vanish asymptotically:

\[
\lim_{J \to \infty} \lim_{n \to \infty} \sup \|w^J_n\|_{(L_t^\infty L_x^p \cap L_t^3 L_x^6)(\mathbb{R} \times \mathbb{R}^3)} = 0 \quad \forall 2 < p < 6
\]
- orthogonality of the energy:

\[
\|\tilde{u}_n\|_{\mathcal{H}}^2 = \sum_{1 \leq j < J} \|\tilde{v}^j\|_{\mathcal{H}}^2 + \|\tilde{w}^J_n\|_{\mathcal{H}}^2 + o(1)
\]
We can extract further profiles from the Strichartz sea if $w_n^4$ does not vanish as $n \to \infty$ in a suitable sense. In the radial case this means $\lim_{n \to \infty} \|w_n^4\|_{L_t^\infty L_x^p(\mathbb{R}^3)} > 0$. 

Profiles and Strichartz sea
Lorentz transformations

\[
\begin{bmatrix}
    t' \\
    x'_1 \\
    x'_2 \\
    x'_3
\end{bmatrix} =
\begin{bmatrix}
    \cosh \alpha & \sinh \alpha & 0 & 0 \\
    \sinh \alpha & \cosh \alpha & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    t \\
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix}
\]

**Figure:** Causal structure of space-time
Further remarks on Bahouri-Gérard

- Noncompact symmetry groups: space-time translations and Lorentz transforms.

  Compact symmetry groups: Rotations. Lorentz transforms do not appear in the profiles: Energy bound compactifies them.

- Dispersive error $w_n^{J}$ is not an energy error!

- In the radial case only need time translations
The focusing NLKG equation

The focusing NLKG
\[ \Box u + u = \partial_{tt} u - \Delta u + u = u^3 \]

has indefinite conserved energy
\[ E(u, \dot{u}) = \int_{\mathbb{R}^3} \left( \frac{1}{2} |\dot{u}|^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |u|^2 - \frac{1}{4} |u|^4 \right) dx \]

- Local wellposedness for $H^1 \times L^2(\mathbb{R}^3)$ data
- Small data: global existence and scattering
- Finite time blowup $u(t) = \sqrt{2}(T - t)^{-1}(1 + o(1))$ as $t \to T^-$
  Cutoff to a cone using finite propagation speed to obtain finite energy solution.
- Stationary solutions $-\Delta \varphi + \varphi = \varphi^3$, ground state $Q(r) > 0$
Payne-Sattinger theory; saddle structure of energy near \( Q \)

Criterion: finite-time blowup/global existence?
Yes, provided the energy is less than the ground state energy Payne-Sattinger 1975.

\[
J := J(u) > J(Q) =: J_0
\]

\[
K := K(u) > K(Q) =: K_0
\]

\[
J(\phi) = \int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla \phi|^2 + \frac{1}{2} |\phi|^2 - \frac{1}{4} |\phi|^4 \right) dx
\]

\[
K(\phi) = \int_{\mathbb{R}^3} \left( |\nabla \phi|^2 + |\phi|^2 - |\phi|^4 \right) dx
\]

**Uniqueness of \( Q \) is the foundation!**
Payne-Sattinger theory

\[ j_\varphi(\lambda) := J(e^\lambda \varphi), \varphi \neq 0 \text{ fixed.} \]

```
Figure: Payne-Sattinger well
```

Normalize so that \( \lambda_* = 0 \). Then \( \partial_\lambda j_\varphi(\lambda) \big|_{\lambda=\lambda_*} = K(\varphi) = 0 \).

“Trap” the solution in the well on the left-hand side: need \( E < \inf \{ j_\varphi(0) \mid K(\varphi) = 0, \varphi \neq 0 \} = J(Q) \) (lowest mountain pass). Expect global existence in that case.
Above the ground state energy

Theorem (Nakanishi-S. 2010)
Let $E(u_0, u_1) < E(Q, 0) + \varepsilon^2, (u_0, u_1) \in \mathcal{H}_{\text{rad}}.$ In $t \geq 0$ for NLKG:

1. finite time blowup
2. global existence and scattering to 0
3. global existence and scattering to $Q$: $u(t) = Q + v(t) + o_{H^1}(1)$ as $t \to \infty,$ and $\dot{u}(t) = \dot{v}(t) + o_{L^2}(1)$ as $t \to \infty,$ $\Box v + v = 0,$ $(v, \dot{v}) \in \mathcal{H}.$

All 9 combinations of this trichotomy allowed as $t \to \pm \infty.$

- Applies to $\dim = 3, |u|^{p-1}u,$ $7/3 < p < 5,$ or $\dim = 1, p > 5.$
- Third alternative forms the center stable manifold associated with $(\pm Q, 0).$ Linearized operator $L_+ = -\Delta + 1 - 3Q^2$ has spectrum $\{-k^2\} \cup [1, \infty)$ on $L^2_{\text{rad}}(\mathbb{R}^3).$ Gap $[0, 1)$ difficult to verify, Costin-Huang-S., 2011.
- $\exists$ 1-dim. stable, unstable manifolds at $(\pm Q, 0).$ Stable manifolds:
  Duyckaerts-Merle, Duyckaerts-Holmer-Roudenko 2009
The invariant manifolds

Figure: Stable, unstable, center-stable manifolds
Hyperbolic dynamics near $\pm Q$

Linearized operator $L_+ = -\Delta + 1 - 3Q^2$

- $\langle L_+ Q | Q \rangle = -2\|Q\|_4^4 < 0$
- $L_+ \rho = -k^2 \rho$ unique negative eigenvalue, no kernel over radial functions
- Gap property: $L_+$ has no eigenvalues in $(0, 1]$, no threshold resonance (delicate!) Use Kenji Yajima’s $L^p$-boundedness for wave operators.

Plug $u = Q + v$ into cubic NLKG:

$$\ddot{v} + L_+ v = N(Q, v) = 3Qv^2 + v^3$$

Rewrite as a Hamiltonian system:

$$\partial_t \begin{pmatrix} \dot{v} \\ \dot{\nu} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -L_+ & 0 \end{pmatrix} \begin{pmatrix} \dot{v} \\ \dot{\nu} \end{pmatrix} + \begin{pmatrix} 0 \\ N(Q, v) \end{pmatrix}$$

Then $\text{spec}(A) = \{k, -k\} \cup i[1, \infty) \cup i(-\infty, -1]$ with $\pm k$ simple evals. Formally:

$X_s = P_1 L^2, X_u = P_{-1} L^2, X_c$ is the rest.
Figure: Spectrum of nonselfadjoint linear operator in phase space
Numerical 2-dim section through $\partial S_+$ (with R. Donninger)

Figure: $(Q + Ae^{-r^2}, Be^{-r^2})$

- soliton at $(A, B) = (0, 0)$, $(A, B)$ vary in $[-9, 2] \times [-9, 9]$
- **RED**: global existence, **WHITE**: finite time blowup, **GREEN**: $\mathcal{PS}_+$, **BLUE**: $\mathcal{PS}_-$
- Our results apply to a neighborhood of $(Q, 0)$, boundary of the red region looks smooth (caution!)
Variational structure above $E(Q, 0)$

$E := E(u, u) > J(Q) + \varepsilon^2 =: J$

- Solution can pass through the balls. Energy is no obstruction anymore as in the Payne-Sattinger case.

- Key to description of the dynamics: One-pass (no return) theorem. The trajectory can make only one pass through the balls.

- Point: Stabilization of the sign of $K(u(t))$. 
One-pass theorem (non-perturbative)

Figure: Possible returning trajectories

Such trajectories are excluded by means of an indirect argument using a variant of the virial argument that was essential to the rigidity step of concentration compactness.
One-pass theorem

**Crucial no-return property:** Trajectory does **not return to balls around** \((\pm Q, 0)\). Suppose it did; Use **virial identity**

\[
\partial_t \langle w \dot{u} | Au \rangle = -\int_{\mathbb{R}^3} (|\nabla u|^2 - \frac{3}{4}|u|^4) \, dx + \text{error}, \quad A = \frac{1}{2} (x \nabla + \nabla x)
\]

where \(w = w(t, x)\) is a **space-time cutoff** that lives on a **rhombus**, and the “error” is controlled by the **external energy**.

Finite propagation speed \(\Rightarrow\) error controlled by **free energy outside large balls** at times \(T_1, T_2\).
Integrating between \(T_1, T_2\) gives **contradiction**; the **bulk** of the integral of \(K_2(u(t))\) here comes from **exponential ejection mechanism** near \((\pm Q, 0)\).

**Non-perturbative argument.**
Figure: Space-time cutoff for the virial identity
Open problem

Complete description of possible long-term dynamics: Given focusing NLKG3 in $\mathbb{R}^3$ with radial energy data, show that the solution either

- blows up in finite time
- exists globally, scatters to one of the stationary solutions $-\Delta \varphi + \varphi = \varphi^3$ (including 0)

Moreover, describe dynamics, center-stable manifolds associated with $\varphi$.

Evidence: With dissipation given by $\alpha \partial_t u$ term, result holds (Burq-Raugel-S.).

Critical equation: $\Box u = u^5$ in $\mathbb{R}^3$, Duyckaerts-Kenig-Merle proved analogous result with rescaled ground-state profiles $\sqrt{\lambda} W(\lambda x)$, $W(x) = (1 + |x|^2/3)^{-\frac{1}{2}}$.

Obstruction: Exterior energy estimates in DKM scheme fail in the KG case due to speed of propagation $< 1$. 
Equivariant wave maps

\( u : \mathbb{R}^{1+2}_{t,x} \to S^2 \) satisfies WM equation

\[ \Box u \perp T_u S^2 \iff \Box u = u(|\partial_t u|^2 - |\nabla u|^2) \]

as well as equivariance assumption \( u \circ R = R \circ u \) for all \( R \in SO(2) \)

Figure: Equivariance and Riemann sphere
Equivariant wave maps

\[ u(t, r, \phi) = (\psi(t, r), \phi), \text{ spherical coordinates, } \psi \text{ angle from north pole satisfies} \]

\[ \psi_{tt} - \psi_{rr} - \frac{1}{r} \psi_r + \frac{\sin(2\psi)}{2r^2} = 0, \quad (\psi, \psi_t)(0) = (\psi_0, \psi_1) \]

- Conserved energy

\[ E(\psi, \psi_t) = \int_0^\infty \left( \psi_t^2 + \psi_r^2 + \frac{\sin^2(\psi)}{r^2} \right) r \, dr \]

- \( \psi(t, \infty) = n\pi, n \in \mathbb{Z} \), homotopy class = degree = \( n \)

- Stationary solutions = harmonic maps = \( 0, \pm Q(r/\lambda) \), where \( Q(r) = 2 \arctan r \). This is the identity \( \mathbb{S}^2 \to \mathbb{S}^2 \) with stereographic projection onto \( \mathbb{R}^2 \) as domain (conformal map!).
Large data results for equivariant wave maps

Theorem (Côte, Kenig, Lawrie, S. 2012)
Let \((\psi_0, \psi_1)\) be smooth data.

1. Let \(E(\psi_0, \psi_1) < 2E(Q, 0)\), degree 0. Then the solution exists globally, and scatters (energy on compact sets vanishes as \(t \to \infty\)). For any \(\delta > 0\) there exist data of energy < \(2E(Q, 0) + \delta\) which blow up in finite time.

2. Let \(E(\psi_0, \psi_1) < 3E(Q, 0)\), degree 1. If the solution \(\psi(t)\) blows up at time \(t = 1\), then there exists a continuous function, \(\lambda : [0, 1) \to (0, \infty)\) with \(\lambda(t) = o(1 - t)\), a map \(\vec{\varphi} = (\varphi_0, \varphi_1) \in \mathcal{H}\) with \(E(\vec{\varphi}) = E(\vec{\psi}) - E(Q, 0)\), and a decomposition

\[
\vec{\psi}(t) = \vec{\varphi} + (Q \cdot /\lambda(t)), 0) + \vec{\epsilon}(t) \quad (\star)
\]

s.t. \(\vec{\epsilon}(t) \in \mathcal{H}, \quad \vec{\epsilon}(t) \to 0\) in \(\mathcal{H}\) as \(t \to 1\).
For **degree 1** have an analogous classification to \((\star)\) for **global solutions**.

Côte 2013: bubble-tree classification for **all** energies along a sequence of times.

**Open problems:** (A) all times, rather than a sequence (B) construction of bubble trees.

Duyckaerts, Kenig, Merle 12 established classification results for \(\Box u = u^5\) in \(\dot{H}^1 \times L^2(\mathbb{R}^3)\) with \(W(x) = (1 + |x|^2/3)^{-\frac{1}{2}}\) instead of \(Q\).

Construction of \((\star)\) by Krieger-S.-Tataru 06 in finite time, Donninger-Krieger 13 in **infinite time** (for critical NLW)

Crucial role is played by Michael Struwe’s bubbling off theorem (equivariant): if blowup happens, then there exists a sequence of times approaching blowup time, such that a rescaled version of the wave map approaches locally in energy space a harmonic map of positive energy.
Struwe’s cuspidal energy concentration

Rescalings converge in $L^2_{t,r}$-sense to a stationary wave map of positive energy, i.e., a harmonic map.
\[ \Box u = 0, \quad u(0) = f \in \dot{H}^1(\mathbb{R}^d), \quad u_t(0) = g \in L^2(\mathbb{R}^d) \text{ radial} \]

Duyckaerts-Kenig-Merle 2011: for all \( t \geq 0 \) or \( t \leq 0 \) have \( E_{\text{ext}}(\bar{u}(t)) \geq cE(f, g) \) provided dimension odd. \( c > 0, \ c = \frac{1}{2} \)

Heuristics: incoming vs. outgoing data.
Exterior energy: even dimensions

Côte-Kenig-S. 2012: This **fails in even dimensions**.

\(d = 2, 6, 10, \ldots\) holds for data \((0, g)\) but fails in general for \((f, 0)\).

\(d = 4, 8, 12, \ldots\) holds for data \((f, 0)\) but fails in general for \((0, g)\).

Fourier representation, Bessel transform, dimension \(d\) reflected in the phase of the Bessel asymptotics, computation of the asymptotic exterior energy as \(t \to \pm \infty\).

For our \(3E(Q, 0)\) theorem we need \(d = 4\) result; rather than \(d = 2\) due to repulsive \(\frac{\psi}{r^2}\)-potential coming from \(\frac{\sin(2\psi)}{2r^2}\).

\((f, 0)\) result suffices by Christodoulou, Tahvildar-Zadeh, Shatah results from mid 1990s. Showed that at blowup \(t = T = 1\) have vanishing kinetic energy

\[
\lim_{t \to 1} \frac{1}{1 - t} \int_{t}^{1} \int_{0}^{1-t} |\psi(t, r)|^2 r dr dt = 0
\]

No result for Yang-Mills since it corresponds to \(d = 6\)
Exterior energy: odd dimensions

Duyckaerts-Kenig-Merle: in radial $\mathbb{R}^3$ one has for all $R \geq 0$

$$\max \lim_{t \to \pm \infty} \int_{|x| > t + R} |\nabla_{t,x} u|^2 \, dr \geq c \int_{|x| > R} [(ru)^2 + (ru)^2] \, dr$$

**Note:** RHS is not standard energy! Orthogonal projection perpendicular to Newton potential $(r^{-1}, 0)$ in $H^1 \times L^2(\mathbb{R}^3 : r > R)$.

Kenig-Lawrie-S. 13 noted this projection and extended the exterior energy estimate to $d = 5$: project perpendicular to plane $(\xi r^{-3}, \eta r^{-3})$ in $H^1 \times L^2(\mathbb{R}^5 : r > R)$

Kenig-Lawrie-Liu-S. 14 all odd dimensions, projections off of similar but larger and more complicated linear subspaces.

**Relevance:** Wave maps in $\mathbb{R}^3$ outside of a ball with arbitrary degree of equivariance lead to all odd dimensions.
Wave maps outside a ball

Consider equivariant wave maps from $\mathbb{R}^3 \setminus B(0,1) \to S^3$ with Dirichlet condition at $R = 1$. Supercritical becomes subcritical, easy to obtain global smooth solutions.

Conjecture by Bizon-Chmaj-Maliborski 2011: All smooth solutions scatter to the unique harmonic map in their degree class.

Results:

- **Lawrie-S. 2012**: Proved for degree 0 and asymptotic stability for degree 1. Follows Kenig-Merle concentration compactness approach with rigidity argument carried out by a virial identity (complicated).

- **Kenig-Lawrie-S. 2013**: Proved for all degrees in equivariance class 1. Uses exterior energy estimates instead of virial.

- **Kenig-Lawrie-Liu-S. 2014**: Proved for all degrees and all equivariance classes. Requires exterior energy estimates in all odd dimensions.

Soliton resolution conjecture holds in this case.
THANK YOU FOR YOUR ATTENTION!