Branching to maximal compact subgroups

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Outline

Prequel

Introduction

What are the questions?

Equivariant $K$-theory

$K$-theory and representations

Birthday business
Branching to maximal compact subgroups

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Prequel
Introduction
Questions
K-theory
K-theory & repns
Birthday business

Something to do during the talk

$k_v$ local field, $G_v = G(k_v)$ reductive, $g_v = \text{Lie}(G_v)$.

$g_v^* = \text{lin fnls on } g_v$, $O_v = G_v \cdot x_v$ coadjt orbit.

$N(O_v) = \text{def } k_v \cdot O_v \cap N_v^*$ asymp nilp cone of $O_v$.

$k$ global, $\pi = \bigotimes_v \pi_v$ automorphic rep of $G$ reductive.

Conjecture

1. $\exists$ coadjt orbit $G(k) \cdot x \subset g(k)^*$, $N(G_v \cdot x) = \text{WF}(\pi_v)$.
2. $\exists$ global version of local char expansions for $\pi_v$.

Says $G(k) \cdot x \rightsquigarrow$ asymp of $K$-types at each place.

$O_k = \text{def } G(k) \cdot x \rightsquigarrow N(O_k) = \bar{k} \cdot O_k \cap N_k^*$

$N(O_k) =$ closure of one nilp orbit $\mathcal{M}$.

$N(G_v \cdot x)$ contained in $N(O_k)$, but may not meet $\mathcal{M}$. 
Setting

Compact groups $K$ are relatively easy... Noncompact groups $G$ are relatively hard.

Harish-Chandra et al. idea:

$$\text{understand } \pi \in \hat{G} \iff \text{understand } \pi|_K$$

(nice compact subgroup $K \subset G$).

Get an invariant of a repn $\pi \in \hat{G}$:

$$m_\pi : \hat{K} \to \mathbb{N}, \quad m_\pi(\mu) = \text{mult of } \mu \text{ in } \pi|_K.$$ 

1. What’s the support of $m_\pi$? (subset of $\hat{K}$)
2. What’s the rate of growth of $m_\pi$?
3. What functions on $\hat{K}$ can be $m_\pi$?
Examples

1. \( G = GL(n, \mathbb{C}), K = U(n) \). Typical restriction to \( K \) is
   \[
   \pi|_K = \text{Ind}^{U(n)}_{U(1)^n}(\gamma) = \sum_{\mu \in \hat{U}(n)} m_\mu(\gamma) \gamma \quad (\gamma \in \hat{U(1)^n}) : 
   \]
   \[
   m_\pi(\mu) = \text{mult of } \mu \text{ in } \pi \text{ is } m_\mu(\gamma) = \text{dim of } \gamma \text{ wt space.}
   \]

2. \( G = GL(n, \mathbb{R}), K = O(n) \). Typical restriction to \( K \) is
   \[
   \pi|_K = \text{Ind}^{O(n)}_{O(1)^n}(\gamma) = \sum_{\mu \in \hat{O}(n)} m_\mu(\gamma) : 
   \]
   \[
   m_\pi(\mu) = \text{mult of } \mu \text{ in } \pi \text{ is } m_\mu(\gamma) = \text{mult of } \gamma \text{ in } \mu.
   \]

3. \( G \) split of type \( E_8, K = Spin(16) \). Typical res to \( K \) is
   \[
   \pi|_{Spin(16)} = \text{Ind}^{Spin(16)}_M(\gamma) = \sum_{\mu \in \hat{Spin(16)}} m_\mu(\gamma) \gamma;
   \]
   here \( M \subset Spin(16) \) subgp of order 512, cent ext of \((\mathbb{Z}/2\mathbb{Z})^8\).

Moral: may compute \( m_\pi \) using compact groups.
Machinery to use

Roger’s approach to these questions:

Roger’s results on classical groups

Our approach today:

Use fundamental tools

Ask George and Roman for advice

Get new results on general groups
Plan for today

Work with real reductive Lie group $G(\mathbb{R})$.

Describe (old) associated cycle $\mathcal{AC}(\pi)$ for irr rep $\pi \in \hat{G}(\mathbb{R})$: geometric shorthand for approximating restriction to $K(\mathbb{R})$ of $\pi$.

Describe (new) algorithm for computing $\mathcal{AC}(\pi)$.

A real algorithm is one that’s been implemented on a computer. This one has not, but should be possible soon.
Assumptions

\( G(\mathbb{C}) = G = \text{cplx conn reductive alg gp.} \)
\( G(\mathbb{R}) = \text{group of real points for a real form.} \)
Could allow fin cover of open subgp of \( G(\mathbb{R}) \), so allow nonlinear.
\( K(\mathbb{R}) \subset G(\mathbb{R}) \) max cpt subgp; \( K(\mathbb{R}) = G(\mathbb{R})^\theta. \)
\( \theta = \text{alg inv of } G; \ K = G^\theta \) possibly disconn reductive.

Harish-Chandra idea:
\( \infty\)-diml reps of \( G(\mathbb{R}) \leftrightarrow \text{alg gp } K \curvearrowright \text{cplx Lie alg } \mathfrak{g} \)
\((\mathfrak{g}, K)\)-module is vector space \( V \) with
1. repn \( \pi_K \) of algebraic group \( K \): \( V = \sum_{\mu \in \hat{K}} m_V(\mu)\mu \)
2. repn \( \pi_\mathfrak{g} \) of cplx Lie algebra \( \mathfrak{g} \)
3. \( d\pi_K = \pi_\mathfrak{g}|_t, \quad \pi_K(k)\pi_\mathfrak{g}(X)\pi_K(k^{-1}) = \pi_\mathfrak{g}(\text{Ad}(k)X). \)

In module notation, cond (3) reads \( k \cdot (X \cdot \nu) = (\text{Ad}(k)X) \cdot (k \cdot \nu). \)
Geometrizing representations

\( G(\mathbb{R}) \) real reductive, \( K(\mathbb{R}) \) max cpt, \( g(\mathbb{R}) \) Lie alg

\( \mathcal{N}^* = \text{cone of nilpotent elements in } g^* \).

\( \mathcal{N}^*_R = \mathcal{N}^* \cap i g(\mathbb{R})^* \), finite \# \( G(\mathbb{R}) \) orbits.

\( \mathcal{N}^*_\theta = \mathcal{N}^* \cap (g/\mathfrak{t})^* \), finite \# \( K \) orbits.

**Goal 1:** Attach orbits to representations in theory.

**Goal 2:** Compute them in practice.

"In theory there is no difference between theory and practice. In practice there is." Jan L. A. van de Snepscheut (or not).

\((\pi, \mathcal{H}_\pi)\) irr rep of \( G(\mathbb{R}) \) \hspace{1cm} \( \mathcal{H}_\pi^K \) irr \( (g, K) \)-module

\( \downarrow \) Howe wavefront \hspace{1cm} \( \downarrow \) assoc var of gr

\( WF(\pi) = G(\mathbb{R}) \text{ orbs on } \mathcal{N}^*_R \) \hspace{1cm} \( \mathcal{H}C(\pi) = K \text{ orbs on } \mathcal{N}^*_\theta \)

Columns related by HC, Kostant-Rallis, Sekiguchi, Schmid-Vilonen.

So **Goal 1** is completed. Turn to **Goal 2**...
Associated varieties

\[ \mathcal{F}(g, K) = \text{finite length } (g, K)\text{-modules}. \ldots \]

noncommutative world we care about.

\[ C(g, K) = \text{f.g. } (S(g/\mathfrak{t}), K)\text{-modules, support } \subset \mathcal{N}_\theta^*. \ldots \]

commutative world where geometry can help.

\[ \mathcal{F}(g, K) \xrightarrow{\text{gr}} C(g, K) \]

\text{gr not quite a functor (choice of good filts), but}

\textbf{Prop.} \text{gr induces surjection of Grothendieck groups}

\[ K\mathcal{F}(g, K) \xrightarrow{\text{gr}} KC(g, K); \]

image records restriction to \( K \) of HC module.

So restrictions to \( K \) of HC modules sit in equivariant coherent sheaves on nilp cone in \((g/\mathfrak{t})^*\)

\[ KC(g, K) =_{\text{def}} K^K(\mathcal{N}_\theta^*), \]

equivariant \( K \)-theory of the \( K \)-nilpotent cone.

\textbf{Goal 2:} compute \( K^K(\mathcal{N}_\theta^*) \) and the map \textbf{Prop.}
Equivariant $K$-theory

**Setting:** (complex) algebraic group $K$ acts on (complex) algebraic variety $X$.

Originally $K$-theory was about vector bundles, but for us coherent sheaves are more useful.

$\text{Coh}^K(X) = \text{abelian categ of coh sheaves on } X \text{ with } K \text{ action.}$

$K^K(X) = \text{def Grothendieck group of } \text{Coh}^K(X)$.

**Example:** $\text{Coh}^K(\text{pt}) = \text{Rep}(K) \text{ (fin-diml reps of } K)$.  

$K^K(\text{pt}) = R(K) = \text{rep ring of } K \text{; free } \mathbb{Z}-\text{module, basis } \hat{K}$.

**Example:** $X = K/H; \text{Coh}^K(K/H) \simeq \text{Rep}(H)$

$E \in \text{Rep}(H) \rightsquigarrow \mathcal{E} = \text{def } K \times_H E \text{ eqvt vector bdle on } K/H$

$K^K(K/H) = R(H)$.  

**Example:** $X = V \text{ vector space}$.  

$E \in \text{Rep}(K) \rightsquigarrow \text{proj module } O_V(E) = \text{def } O_V \otimes E \in \text{Coh}^K(X)$

proj resolutions $\implies K^K(V) \simeq R(K), \text{ basis } \{O_V(\tau)\}$.  


Doing nothing carefully

Suppose $K \rtimes X$ with finitely many orbits:

$X = Y_1 \cup \cdots \cup Y_r, \quad Y_i = K \cdot y_i \cong K/Ky_i$.

Orbits partially ordered by $Y_i \geq Y_j$ if $Y_j \subset Y_i$.

$(\tau, E) \in K^{|Y_i|} \leadsto \mathcal{E}(\tau) \in \text{Coh}^K(Y_i)$.

Choose (always possible) $K$-equivariant coherent extension

$\tilde{\mathcal{E}}(\tau) \in \text{Coh}^K(\overline{Y_i}) \leadsto [\tilde{\mathcal{E}}] \in K^K(\overline{Y_i})$.

Class $[\tilde{\mathcal{E}}]$ on $\overline{Y_i}$ unique modulo $K^K(\partial Y_i)$.

Set of all $[\tilde{\mathcal{E}}(\tau)]$ (as $Y_i$ and $\tau$ vary) is basis of $K^K(X)$.

Suppose $M \in \text{Coh}^K(X)$; write class of $M$ in this basis

$[M] = \sum_{i=1}^r \sum_{\tau \in K^{|Y_i|}} n_\tau(M)[\tilde{\mathcal{E}}(\tau)]$.

Maxl orbits in $\text{Supp}(M) = \text{maxl } Y_i$ with some $n_\tau(M) \neq 0$.

Coeffs $n_\tau(M)$ on maxl $Y_i$ ind of choices of exts $\tilde{\mathcal{E}}(\tau)$. 
Our story so far

We have found

1. homomorphism

\[ \text{virt } G(\mathbb{R}) \text{ reps } K\mathcal{F}(g, K) \xrightarrow{\text{gr}} K^K(N_\theta^*) \text{ eqvt } K\text{-theory} \]

2. geometric basis \( \{[\mathcal{E}(\tau)]\} \) for \( K^K(N_\theta^*) \), indexed by irr reps of isotropy gps

3. expression of \([\text{gr}(\pi)]\) in geom basis \( \xrightarrow{\sim} \mathcal{A}C(\pi) \).

Problem is expressing ourselves...

Teaser for the next section: Kazhdan and Lusztig taught us how to express \( \pi \) using std reps \( I(\gamma) \):

\[ [\pi] = \sum_{\gamma} m_{\gamma}(\pi)[I(\gamma)], \quad m_{\gamma}(\pi) \in \mathbb{Z}. \]

\( \{[\text{gr } I(\gamma)]\} \) is another basis of \( K^K(N_\theta^*) \).

Last goal is compute change of basis matrix.
The last goal

Studying cone $\mathcal{N}_\theta^* = \text{nilp lin functionals on } g/\mathfrak{k}$.

Found (for free) basis $\{[\mathcal{E}(\tau)]\}$ for $K^K(\mathcal{N}_\theta^*)$, indexed by orbit $K/K^i$ and irr rep $\tau$ of $K^i$.

Found (by rep theory) second basis $\{[\text{gr } l(\gamma)]\}$, indexed by (parameters for) std reps of $G(\mathbb{R})$.

To compute associated cycles, enough to write

$$[\text{gr } l(\gamma)] = \sum_{\text{orbits}} \sum_{\tau \text{ irr for isotropy}} N_\tau(\gamma)[\mathcal{E}(\tau)].$$

Equivalent to compute inverse matrix

$$[\mathcal{E}(\tau)] = \sum_\gamma n_\gamma(\tau)[\text{gr } l(\gamma)].$$

Need to relate geom of nilp cone to geom std reps: parabolic subgroups. Use Springer resolution.
Introducing Springer

$g = \mathfrak{t} \oplus \mathfrak{s}$ Cartan decomp, $\mathcal{N}_\theta^* \cong \mathcal{N}_\theta = \text{def } \mathcal{N} \cap \mathfrak{s}$ nilp cone in $\mathfrak{s}$.

Kostant-Rallis, Jacobson-Morozov: nilp $X \in \mathfrak{s} \mapsto Y \in \mathfrak{s}$, $H \in \mathfrak{t}$

$[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$,

$g[k] = \mathfrak{t}[k] \oplus \mathfrak{s}[k]$ (ad($H$) eigenspace).

$\mapsto g[\geq 0] = \text{def } q = 1 + u$ $\theta$-stable parabolic.

**Theorem** (Kostant-Rallis) Write $O = K \cdot X \subset \mathcal{N}_\theta$.

1. $\mu: O_Q = \text{def } K \times_{Q \cap K} \mathfrak{s}[\geq 2] \to \overline{O}$, $(k, Z) \mapsto \text{Ad}(k)Z$ is proper birational map onto $\overline{O}$.

2. $K^X = (Q \cap K)^X = (L \cap K)^X(U \cap K)^X$ is a Levi decomp; so $\overline{K^X} = [(L \cap K)^X]$.

So have resolution of singularities of $\overline{O}$:

$K \times_{Q \cap K} \mathfrak{s}[\geq 2] \xrightarrow{\text{vec bdle}} K/Q \cap K \xrightarrow{\mu} \overline{O}$

Use it (i.e., copy McGovern, Achar) to calculate equivariant $K$-theory...
Using Springer to calculate $K$-theory

$X \in \mathcal{N}_\theta$ represents $O = K \cdot X$.

$\mu: O_Q =_{df} K \times_{Q \cap K} \mathbb{C}[\geq 2] \to \overline{O}$ Springer resolution.

**Theorem** Recall $\widehat{KX} = [(L \cap K)^X]$.

1. $K^K(O_Q)$ has basis of eqvt vec bdles:
   $$(\sigma, F) \in \text{Rep}(L \cap K) \leadsto F(\sigma).$$

2. Get extension of $E(\sigma|_{(L \cap K)^X})$ on $O$
   $$[\overline{F}(\sigma)] =_{df} \sum_i (-1)^i [R^i \mu_*(F(\sigma))] \in K^K(\overline{O}).$$

3. Compute (very easily) $[\overline{F}(\sigma)] = \sum_\gamma n_\gamma(\sigma)[\text{gr} l(\gamma)]$.

4. Each irr $\tau \in [(L \cap K)^X]$ extends to (virtual) rep $\sigma(\tau)$ of $L \cap K$; can choose $\overline{F}(\sigma(\tau))$ as extension of $E(\tau)$. 
Now we’re done

Recall $X \in N_\theta \leadsto O = K \cdot X$; $\tau \in [(L \cap K)^X]^\sim$. Now we know formulas

$$[\widetilde{E}(\tau)] = [\mathcal{F}(\sigma(\tau))] = \sum_{\gamma} n_\gamma(\tau)[\text{gr } I(\gamma)].$$

Here’s why this does what we want:

1. inverting matrix $n_\gamma(\tau) \leadsto$ matrix $N_\tau(\gamma)$ writing $[\widetilde{E}(\tau)]$ in terms of $[\text{gr } I(\gamma)]$.

2. multiplying $N_\tau(\gamma)$ by Kazhdan-Lusztig matrix $m_\gamma(\pi)$ $\leadsto$ matrix $n_\tau(\pi)$ writing $[\text{gr } \pi]$ in terms of $[\widetilde{E}(\tau)]$.

3. Nonzero entries $n_\tau(\pi) \leadsto A\mathcal{C}(\pi)$.

Side benefit: algorithm (for $G(\mathbb{R})$ cplx) also computes bijection (conj by Lusztig, estab by Bezrukavnikov)

$$(\text{dom wts}) \leftrightarrow (\text{pairs } (\tau, O))$$
Mirror, mirror, on the wall

Who’s the fairest one of all?

The winner and still champion!