Theta Correspondence for Dummies

( Correspondance Theta pour les nuls )

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**Theta Correspondence**

\[ Mp = \tilde{Sp}(2n, F): \text{ metaplectic group} \]

\((G, G')\) a reductive dual pair:

\[ G = \text{Cent}_{Mp}(G'), \quad G' = \text{Cent}_{Mp}(G) \]

\(\psi\) character of \(F\), \(\rightarrow\) oscillator representation \(\omega = \omega_\psi\)

**Definition:** \(\pi \in \hat{G}, \pi' \in \hat{G}', \text{ say } \pi \leftrightarrow \pi' \text{ if} \)

\[ \text{Hom}_{G \times G'}(\omega, \pi \boxtimes \pi') \neq 0 \]

**Howe Duality Theorem** (Howe, Waldspurger, Gan-Takeda)

\(F\) local

\(\pi \leftrightarrow \pi' \text{ is a bijection} \)

(between subsets of \(\hat{G}\) and \(\hat{G}'\))

**Definition:** \(\pi' = \theta(\pi), \pi = \theta(\pi')\)
Describe $\pi \to \theta(\pi)$ (in terms of some kinds of parameters)

Properties of the map: preserving tempered, unitary, relation on wave front sets, functoriality (Langlands/Arthur)...

Typically there are some easy cases, and some hard ones

\[
\pi \text{ } Sp(2m, F) \xrightarrow{\theta} \begin{cases}
O(2n + 4) & \theta_{m,2n+4}(\pi) = \text{non-tempered} \\
O(2n + 2) & \theta_{m,2n+2}(\pi) = \text{non-tempered} \\
O(2n) & \theta_{m,2n}(\pi) = \text{discrete series} \\
O(2n - 2) & \theta_{m,2n-2}(\pi) = \text{discrete series} \\
O(2n - 4) & \text{0}
\end{cases}
\]
\( \Theta(\pi) \)

\( \theta(\pi) \) irreducible, \( \omega \rightarrow \pi \boxtimes \theta(\pi) \)

**Definition** (Howe) \( \omega(\pi) \) = the maximal \( \pi \)-isotypic quotient of \( \omega \)

\( \Theta(\pi) \) (“big-theta” of \( \pi \)):

\[
\omega(\pi) \simeq \pi \boxtimes \Theta(\pi)
\]

Proof of the duality theorem: \( \theta(\pi) \) is the unique irreducible quotient of \( \Theta(\pi) \)

Generically, \( \Theta(\pi) \) is irreducible and \( \theta(\pi) = \Theta(\pi) \)
The structure of $\Theta(\pi)$

$\Theta(\pi)$ is important, interesting, complicated

$\Theta(1)$ (Kudla, Rallis, . . .)

Structure of reducible principal series (Howe. . .)

Lee/Zhu: $Sp(2n, \mathbb{R})$:
Example: see-saw pairs and reciprocity

Howe

\[
\begin{array}{c}
\Theta(\sigma') & \Theta(\pi) \\
\pi & \sigma'
\end{array}
\]

\[
\Theta(\sigma')[\pi] \boxtimes \sigma' \simeq \pi \boxtimes \Theta(\pi)[\sigma']
\]

Roughly:

\[
mult_G(\pi, \Theta(\sigma')) = mult_{H'}(\sigma', \Theta(\pi))
\]
Theta correspondence and induction

\[ \Theta_{m,n+r} : \text{GL}(m) \rightarrow \text{GL}(n) \]

\[ \Theta_{m,n} : \text{GL}(n + r) \rightarrow \text{GL}(n) \]

\[ m = n: \Theta_{n,n}(\pi) = \theta_{n,n}(\pi) = \pi^* \]

Kudla: \( P = MN, \quad M = \text{GL}(n) \times \text{GL}(r) \)

\[ \text{Hom}_{\text{GL}(m)}(\omega_{m,n+r}, \pi \boxtimes \text{Ind}_P^{\text{GL}(n+r)}(\theta_{m,n}(\pi) \otimes 1)) \neq 0 \]

\[ \text{Ind}_P^{\text{GL}(n+r)}(\theta_{m,n}(\pi) \otimes 1) \]
$\Theta_{m,n+r}(\pi) \equiv \text{Ind}_{P}^{GL(n+r)}(1 \otimes \theta_{m,n}(\pi))$

$\theta_{m,n+r}(\pi)$ is (?) the unique irreducible quotient of

$\text{Ind}_{P}^{GL(n+r)}(1 \otimes \theta_{m,n}(\pi))$

Neither is true in general

$\omega = S(M_{m,n})$

filtration: $\omega_k$: functions supported on matrices of rank $\geq k$:

$0 = \omega_t \subset \omega_{t-1} \subset \cdots \subset \omega_0 = \omega$

Serious issues with extensions here... also reducibility of induced representations
Basic Principle

$$\text{Hom} \to \text{Ext} \to \text{EP} = \sum_i (-1)^i \text{Ext}^i$$

(+ vanishing. . . )

Problem: Study

$$\text{Ext}^i_{G \times G'}(\omega, \pi \boxtimes \pi'), \text{EP}_{G \times G'}(\omega, \pi \boxtimes \pi')$$

alternatively:

$$\text{Ext}^i_G(\omega, \pi), \text{EP}_G(\omega, \pi)$$ as (virtual) representations of $G'$

Idea: $\text{EP}_G(\omega, \pi)$ is like $\text{Hom}_G(\omega, \pi)$ with everything made completely reducible. . . all “boundary terms” vanish
Example: GL(1), or Tate’s Thesis

\((G, G') = (GL(1), GL(1)) \subset SL(2, F)\)

\(\omega: S(F) (S = C_c, \text{the Schwarz space})\)

\(\omega(g, h)(f)(x) = f(g^{-1}xh) \text{ (up to } |\det|^\pm\frac{1}{2}\text{)}\)

\(\chi \text{ character of } GL(1)\)

**Question:** \(\text{Hom}_{GL(1)}(S(F), \chi) = ?\)

\[
0 \rightarrow S(F^\times) \rightarrow S(F) \rightarrow \mathbb{C} \rightarrow 0
\]

\(\text{Hom}(, \chi) = \text{Hom}_{GL(1)}(, \chi)\)

\[
0 \rightarrow \text{Hom}(\mathbb{C}, \chi) \rightarrow \text{Hom}(S(F), \chi) \rightarrow \text{Hom}(S(F^\times), \chi) \rightarrow \text{Ext}(\mathbb{C}, \chi)
\]
Example: GL(1), or Tate's Thesis

\[ 0 \rightarrow \text{Hom}(\mathbb{C}, \chi) \rightarrow \text{Hom}(S(F), \chi) \rightarrow \text{Hom}(S(F^\times), \chi) \rightarrow \text{Ext}(\mathbb{C}, \chi) \]

\( \chi \neq 1: \)

\[ 0 \rightarrow 0 \rightarrow \text{Hom}(S(F), \chi) \rightarrow \text{Hom}(S(F^\times), \chi) \rightarrow 0 \]

\( \text{Hom}_{GL(1)}(S(F), \chi) = \text{Hom}_{GL(1)}(S(F^\times), \chi) = \mathbb{C} \)

\( \chi = 1: \)

\[ 0 \rightarrow \mathbb{C} \rightarrow \text{Hom}(S(F), \chi) \rightarrow \text{Hom}(S(F^\times), \chi) \rightarrow \mathbb{C} \rightarrow \text{Ext}^1(S(F), \mathbb{C}) = 0 \]

\( \text{Hom}_{GL(1)}(S(F), \chi) = 1 \) in all cases

Remark: Tate's thesis: this is true provided \( |\chi(x)| = |x|^s \) with \( s > 1 \). General case: analytic continuation in \( \chi \) of Tate L-functions.
Punch line:

**Theorem** (Adams/Prasad/Savin)

Fix $m$, and consider the dual pairs $(G = GL(m), GL(n))$ $n \geq 0$. Let $\pi \in \hat{G}$

$$\operatorname{EP}_G(\omega_{m,n}, \pi)_{\infty} \simeq \begin{cases} 0 & n < m \\ \operatorname{Ind}_P^{GL(n)}(1 \otimes \pi) & n \geq m \end{cases}$$

where $M = GL(n-m) \times GL(m)$

More details...
Euler–Poincare Characteristic

Reference: D. Prasad, *Ext Analogues of Branching Laws*

$F$: $p$-adic field, $G$: reductive group/$F$

$\mathcal{C} = C_G$: category of smooth representations

$S(G) = C_c^\infty(G)$, smooth compactly supported functions, smooth representation of $G \times G$

**Lemma:** $\mathcal{C}$ has enough projectives and injectives

$\text{Ext}^i_G(X, Y)$: derived functors of $\text{Hom}_G(\_, Y)$ or $\text{Hom}_G(X, \_)$. 
Euler-Poincare Characteristic

\[ P = MN \subset G, \text{Ind}_P^G \text{ normalized induction} \quad r_P^G \text{ normalized Jacquet functor} \]

\( X, Y \) smooth

1. \( \text{Ext}_G^i(X, Y) = 0 \) for \( i > \text{split rank of } G \)
2. \( S(G) \) is projective (as a left \( G \)-module)
3. \( \text{Hom}_G(S(G), X)^{G-\infty} \simeq X \)
4. \( \text{EP}_{GL(m)}(X, Y) = 0 \) (\( X, Y \) finite length)
5. \( \text{Ext}_G^i(X, \text{Ind}_P^G(Y)) \simeq \text{Ext}_M^i(r_P^G(X), Y) \)
6. \( \text{Ext}_G^i(\text{Ind}_P^G(X), Y) \simeq \text{Ext}_M^i(X, r_P^G(Y)) \)
7. Kunneth Formula \( (X_1 \text{ admissible}) \):

\[
\text{Ext}^i_{G_1 \times G_2}(X_1 \boxtimes X_2, Y_1 \boxtimes Y_2) \simeq \bigoplus_{j+k=i} \text{Ext}^j_{G_1}(X_1, Y_1) \otimes \text{Ext}^k_{G_2}(X_2, Y_2)
\]
Euler-Poincare Characteristic

\[ X: \ G \times G'\text{-modules (for example: } \omega) \]

\[ Y: \ G\text{-module} \]

\[ \text{Ext}^i_G(X, Y) \text{ is an } G'\text{-module (not necessarily smooth)} \]

**Definition:**

\[ \text{Ext}^i_G(X, Y) \infty = \text{Ext}^i_G(X, Y)^{G'-\infty} \] (a smooth \( G' \)-module)

**Dangerous bend:** \( \text{Ext}^i_G(X, Y) \) is (probably) not the derived functors of \( Y \rightarrow \text{Hom}_G(X, Y)^{G'-\infty} \)

**Definition:** Assume \( \text{Ext}^i_G(X, Y) \) has finite length for all \( i \)

\[ \text{EP}_G(X, Y) = \sum_i (-1)^i \text{Ext}_G(X, Y) \infty \] is a well-defined element of the Grothendieck group of smooth representations of \( G' \)
(\(G, G'\)) dual pair, \(\omega, \pi\) irreducible representation of \(G\)

\(\text{EP}_G(\omega, \pi)\)\(^{\infty}\)

\(\omega \to \pi \boxtimes \Theta(\pi)\)

**Proposition:** \(\text{Hom}_G(\omega, \pi)\)\(^{\infty}\) = \(\Theta(\pi)\)^\(\vee\)

\(\vee\) : smooth dual

**proof:**

\[0 \to \omega[\pi] \to \omega \to \pi \boxtimes \Theta(\pi) \to 0\]

\(\text{Hom}(,\pi)\) is left exact:

\[0 \to \text{Hom}_G(\pi \boxtimes \Theta(\pi), \pi) \to \text{Hom}_G(\omega, \pi) \xrightarrow{\phi} \text{Hom}_G(\omega[\pi], \pi)\]

\(\phi = 0 \Rightarrow \text{Hom}(\omega, \pi) \simeq \Theta(\pi)^*\), take the smooth vectors
Recall: \( \omega_k = S(\text{matrices of rank } \geq k) \)

\[
0 = \omega_t \subset \omega_{t-1} \subset \cdots \subset \omega_0 = \omega
\]

\[
\omega_k / \omega_{k+1} = S(\Omega_k) \quad (\Omega_k = \text{matrices of rank } k)
\]

\[
S(\Omega_k) \cong \text{Ind}_{\text{GL}(k) \times \text{GL}(m-k) \times \text{GL}(k) \times \text{GL}(n-k)}^{\text{GL}(m) \times \text{GL}(n)}(S(\text{GL}(k)) \boxtimes 1).
\]

Compute

\[
\text{Ext}^i_{\text{GL}(m)}(S(\Omega_k), \pi)
\]

By Frobenius reciprocity, Kunneth formula, other basic properties...
\[
\text{Ext}_{\text{GL}(m)}^i(S(\Omega_k), \pi) \simeq \sum_{j=1}^\ell \text{Ind}_{\text{GL}(k) \times \text{GL}(n-k)}^{\text{GL}(n)}(\sigma_j \boxtimes 1) \otimes \text{Ext}_{\text{GL}(m-k)}^i(1, \tau_j)
\]

\[r_{\overline{P}}(\pi) = \sum \sigma_j \boxtimes \tau_j \text{ implies}
\]

**Lemma**

\[\text{Ext}_{\text{GL}(m)}^i(S(\Omega_k), \pi)\] is a finite length \(\text{GL}(n)\)-module

\[\text{EP}_{\text{GL}(m)}(S(\Omega_k), \pi)\] is well defined

\[\text{EP}_{\text{GL}(m)}(S(\Omega_k), \pi) = 0 \text{ unless } k = m.\]
Fix $m$, and consider the dual pairs $(G = GL(m), GL(n))$ $n \geq 0$. 

\[ \pi \in \hat{G} \]

\[ \text{EP}_G(\omega_{m,n}, \pi)^\infty \simeq \begin{cases} 
0 & n < m \\
\text{Ind}_P^{GL(n)}(1 \otimes \pi) & n \geq m
\end{cases} \]

where $M = GL(n - m) \times GL(m)$
Main Theorem: Type I

Similar idea, using Kudla (and MVW) calculation of the Jacquet module of the oscillator representation

For simplicity: state it for \((Sp(2m), O(2n))\) (split orthogonal groups)

\[ \omega = \omega_{m,n} \text{ oscillator representation for } (G, G') = (Sp(2m), O(2n)) \]

\[ t < m \rightarrow M(t) = GL(t) \times Sp(2m - 2t) \subset Sp(2m) \]

\[ P(t) = M(t)N(t), \text{ Ind}_{P(t)}^G() \]

\[ t < n \rightarrow M'(t) = GL(t) \times O(2n - 2t) \subset O(2m) \]

\[ P'(t) = M'(t)N'(t), \text{ Ind}_{P'(t)}^{G'}() \]

\[ \omega_{M(t), M'(t)} \text{ oscillator representation for dual pair } (M(t), M'(t)) \]
Main Theorem: Type I

Theorem Fix an irreducible representation $\pi$ of $M(t)$.

Then

$$\text{EP}_G(\omega_{G,G'}, \text{Ind}_{P(t)}^G(\pi)) \infty \cong \begin{cases} 0 & t > n \\ \text{Ind}_{P'(t)}^{G'}(\text{EP}_{M(t)}(\omega_{M(t),M'(t)}, \pi) \infty) & t \leq n. \end{cases}$$

EP($\omega, _\infty$) commutes with induction