A dynamical system related to GIT

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A gradient system

Let \( \phi \in \mathbb{R}[x_1, \ldots, x_n] \) be a polynomial that is homogeneous of degree \( m \) such that \( \phi(x) \geq 0 \) for all \( x \in \mathbb{R}^n \). We consider the gradient system

\[
\frac{dx}{dt} = -\nabla \phi(x)
\]
A gradient system

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$$\frac{dx}{dt} = -\nabla \phi(x)$$

- Note that

$$\langle \nabla \phi(x), x \rangle = m\phi(x)$$

Denoting by $F(t, x)$ the solution to the system near $t = 0$ with $F(0, x) = x$. Then

$$\frac{d}{dt} \langle F(t, x), F(t, x) \rangle = -2 \langle \nabla \phi(F(t, x)), F(t, x) \rangle$$

$$= -2m\phi(F(t, x)) \leq 0.$$
This implies \( \| F(t, x) \| \leq \| x \| \) where defined for \( t \geq 0 \) and hence \( F(t, x) \) is defined for all \( t \geq 0 \).
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The formula

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\langle \nabla \phi(x), x \rangle = m\phi(x)
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combined with the Schwarz inequality implies that

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\| \nabla \phi(x) \| \| x \| \geq m\phi(x).
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• This implies $\|F(t, x)\| \leq \|x\|$ where defined for $t \geq 0$ and hence $F(t, x)$ is defined for all $t \geq 0$.

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combined with the Schwarz inequality implies that

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• The Lojasiewicz gradient inequality implies the following improvement. There exists $0 < \varepsilon \leq \frac{1}{m-1}$ and $C > 0$ both depending only on $\phi$ such that

$$\|\nabla \phi(x)\|^{1+\varepsilon} \|x\|^{1-(m-1)\varepsilon} \geq C \phi(x).$$
We take $\varepsilon$ and $C$ as above (but allow $\varepsilon = 0$ which is easy). If we write $F$ for $F(t, X)$ and $H(t) = \phi(F(t, x))$ then we have

$$H'(t) = -d\phi(F) \nabla \phi(F) = - \|\nabla \phi(F)\|^2.$$
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$$H'(t) = -d\phi(F) \nabla \phi(F) = -\|\nabla \phi(F)\|^2.$$

If $t \geq 0$ and $\|x\| \leq r$

$$\|\nabla \phi(F)\|^{1+\varepsilon} r^{1-(m-1)\varepsilon} \geq \|\nabla \phi(F)\|^{1+\varepsilon} \|F\|^{1-(m-1)\varepsilon} \geq C\phi(x).$$
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We will now run through what has come to be called “the Lojasiewicz argument” which I learned from a beautiful exposition of Neeman’s theorem by Gerry Schwarz.
\[ \| \nabla \phi(F) \|^{1+\varepsilon} \geq \frac{C}{r^{1-3\varepsilon}} \phi(F). \]
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\[ \| \nabla \phi(F) \|^{2} \geq \left( \frac{C}{r^{1-3\varepsilon}} \right)^{\frac{2}{1+\varepsilon}} \phi(F)^{\frac{2}{1+\varepsilon}}. \]
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\[ |H'(t)| \geq \frac{1}{2} \left( \frac{C}{r^{1-3\varepsilon}} \right)^{\frac{2}{1+\varepsilon}} \phi(F)^{\frac{2}{1+\varepsilon}} = C_1(r)H(t)^{\frac{2}{1+\varepsilon}}. \]
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\[ |H'(t)| \geq \frac{1}{2} \left( \frac{C}{r^{1-3\varepsilon}} \right)^{\frac{2}{1+\varepsilon}} \phi(F)^{\frac{2}{1+\varepsilon}} = C_1(r)H(t)^{\frac{2}{1+\varepsilon}}. \]

Since \( H'(t) \leq 0 \) for \( t \geq 0 \) we have \( -H'(t) \geq C_1(r)H(t)^{\frac{2}{1+\varepsilon}}. \)
Assuming \( H(t) > 0 \) we have
\[ \| \nabla \phi(F) \|^ {1+\varepsilon} \geq \frac{C}{r^{1-3\varepsilon}} \phi(F) . \]

\[ \| \nabla \phi(F) \|^2 \geq \left( \frac{C}{r^{1-3\varepsilon}} \right)^{\frac{2}{1+\varepsilon}} \phi(F)^{\frac{2}{1+\varepsilon}} . \]

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Since \( H'(t) \leq 0 \) for \( t \geq 0 \) we have \( -H'(t) \geq C_1(r) H(t)^{\frac{2}{1+\varepsilon}} \).

Assuming \( H(t) > 0 \) we have

\[ \frac{d}{dt} H(t)^{-\frac{1-\varepsilon}{1+\varepsilon}} = -\frac{1-\varepsilon}{1+\varepsilon} \frac{H'(t)}{H(t)^{\frac{2}{1+\varepsilon}}} \geq C_1(r) \]
\[ H(t)^{-\frac{1-\epsilon}{1+\epsilon}} \geq C_1(r)t. \]
$$H(t)^{-\frac{1-\varepsilon}{1+\varepsilon}} \geq C_1(r)t.$$ 

$$H(t) \leq C_2(r)t^{-\frac{(1+\varepsilon)}{1-\varepsilon}} \leq C_2(r)t^{-(1+\varepsilon)},$$
\[ H(t)^{\frac{1-\epsilon}{1+\epsilon}} \geq C_1(r)t. \]

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This is true if \( H(t) = 0 \) so the formula is valid for all \( t > 0 \).
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This is the first half of the calculus part of the Lojasiewicz argument. The first implication needs only the easy case \( \varepsilon = 0 \). If \( \|x\| \leq r \) then

\[ \phi(F(t, x)) \leq \frac{C(r)}{t} \]

so \( \lim_{t \to +\infty} \phi(F(t, x)) = 0 \) uniformly for \( x \) in compacta. We now do the rest of the Lojasiewicz argument which uses the existence of \( \varepsilon > 0 \).
Let $f(t) = t^{1+\delta}$ with $0 < \delta < \varepsilon$ then for $t > 0$

$$0 < H(t)f'(t) \leq C_2(r)(1 + \delta)t^{-(1-(\varepsilon-\delta))}.$$
Let \( f(t) = t^{1+\delta} \) with \( 0 < \delta < \varepsilon \) then for \( t > 0 \)

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0 < H(t)f'(t) \leq C_2(r)(1 + \delta)t^{-(1-\varepsilon+\delta)}.
\]

\[
H(s)f(s) - H(t)f(t) = \int_t^s \frac{d}{du}(H(u)f(u))du = \int_t^s H(u)f'(u)du + \int_t^s H'(u)f(u)du.
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\[
- \int_t^s H'(u)f(u)du = \int_t^s H(u)f'(u)du + H(t)f(t) - H(s)f(s).
\]

\[
0 \leq H(s)f(s) \leq C_2(r)s^{-(1+\varepsilon)}s^{1+\delta} = C_2(r)s^{-(\varepsilon-\delta)}.
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Let $f(t) = t^{1+\delta}$ with $0 < \delta < \varepsilon$ then for $t > 0$

$$0 < H(t)f'(t) \leq C_2(r)(1 + \delta)t^{-1-(\varepsilon-\delta)}.$$ 

$$H(s)f(s) - H(t)f(t) = \int_t^s \frac{d}{du}(H(u)f(u))du =$$

$$\int_t^s H(u)f'(u)du + \int_t^s H'(u)f(u)du.$$ 

$$-\int_t^s H'(u)f(u)du = \int_t^s H(u)f'(u)du + H(t)f(t) - H(s)f(s).$$

$$0 \leq H(s)f(s) \leq C_2(r)s^{-(1+\varepsilon)}s^{1+\delta} = C_2(r)s^{-(\varepsilon-\delta)}.$$ 

$$\lim_{s \to +\infty} \int_t^s |H'(u)|f(u)du = \int_t^\infty H(u)f'(u)du + H(t)f(t).$$
Thus $\sqrt{|H'(u)| f(u)}$ is in $L^2([t, +\infty))$ for all $t > 0$ and so

$$\sqrt{|H'(u)|} = \sqrt{|H'(u)| f(u) u^{-(1+\delta)/2} \in L^1([t, +\infty))}. $$
Thus \( \sqrt{|H'(u)| f(u)} \) is in \( L^2([t, +\infty)) \) for all \( t > 0 \) and so

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**Theorem.** If \( t > 0 \) then

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\int_t^{+\infty} \left\| \frac{d}{du} F(u, x) \right\| du
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converges uniformly for \( \|x\| \leq r \).
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**Theorem.** If \( t > 0 \) then
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Noting that if $s > t$ then
\[
\int_t^s \frac{d}{du} F(u, x) \, du = F(s, x) - F(t, x)
\]
we have for $t > 0$
\[
\lim_{s \to \infty} F(s, x) = \int_t^{+\infty} \frac{d}{du} F(u, x) \, du + F(t, x).
\]
Finally, set \( L(t, x) = F\left( \frac{t}{1-t}, x \right) \) and define \( L(1, x) \) by the limit above then \( L : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n \) is continuous and since

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\nabla \phi(x) = 0 \iff \phi(x) = 0
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**Theorem.** \( L : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n \) defines a strong deformation retraction of \( \mathbb{R}^n \) onto \( Y = \{ x \in \mathbb{R}^n | \phi(x) = 0 \} \).
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**Corollary.** If \( Z \subset \mathbb{R}^n \) is closed and such that \( F(t, z) \in Z \) for \( t \geq 0 \) and \( z \in Z \) then \( H : [0, 1] \times Z \to Z \) defines a strong deformation retraction of \( Z \) onto \( Z \cap Y \).
Kempf-Ness over the reals

Let $G$ be an open subgroup of a Zariski closed subgroup of $GL(n, \mathbb{R})$ that is closed under real adjoint relative to the standard inner product, $\langle \ldots, \ldots \rangle$, $g \mapsto g^*$. Let $K = G \cap O(n)$. Then $K$ is a maximal compact subgroup of $G$. On $\mathfrak{g} = \text{Lie}(G)$ we put the inner product $\langle X, Y \rangle = \text{tr}(XY^*)$, Set $\mathfrak{p} = \text{Lie}(K)^\perp$ relative to this inner product.
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- We say that an element $v \in \mathbb{R}^n$ is $G$-critical if for any $X \in Lie(G)$, $\langle Xv, v \rangle = 0$. The following is an extension of the Kempf-Ness Theorem first observed by Richardson and Slodoway.
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Theorem. Let $G, K$ be as above. Let $v \in \mathbb{R}^n$. 


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• We say that an element $v \in \mathbb{R}^n$ is $G$-critical if for any $X \in \text{Lie}(G)$, $\langle Xv, v \rangle = 0$. The following is an extension of the Kempf-Ness Theorem first observed by Richardson and Slodoway.

• **Theorem.** Let $G, K$ be as above. Let $v \in \mathbb{R}^n$.

1. If $v$ is critical if and only if $\|gv\| \geq \|v\|$ for all $g \in G$. 

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A dynamical system

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Kempf-Ness over the reals

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**Theorem.** Let $G, K$ be as above. Let $v \in \mathbb{R}^n$.

1. If $v$ is critical if and only if $\|gv\| \geq \|v\|$ for all $g \in G$.
2. If $v$ is critical and if $w \in Gv$ is such that $\|v\| = \|w\|$ then $w \in_kv$. 
Kempf-Ness over the reals

- Let $G$ be an open subgroup of a Zariski closed subgroup of $GL(n, \mathbb{R})$ that is closed under real adjoint relative to the standard inner product, $g \rightarrow g^*$. Let $K = G \cap O(n)$. Then $K$ is a maximal compact subgroup of $G$. On $\mathfrak{g} = \text{Lie}(G)$ we put the inner product $\langle X, Y \rangle = \text{tr}(XY^*)$, Set $\mathfrak{p} = \text{Lie}(K)\perp$ relative to this inner product.

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**Theorem.** Let $G, K$ be as above. Let $v \in \mathbb{R}^n$.

1. If $v$ is critical if and only if $\|gv\| \geq \|v\|$ for all $g \in G$.
2. If $v$ is critical and if $w \in Gv$ is such that $\|v\| = \|w\|$ then $w \in Kv$.
3. If $Gv$ is closed then there exists a critical element in $Gv$. 
Kempf-Ness over the reals

Let $G$ be an open subgroup of a Zariski closed subgroup of $GL(n, \mathbb{R})$ that is closed under real adjoint relative to the standard inner product, $\langle \ldots, \ldots \rangle$, $g \rightarrow g^*$. Let $K = G \cap O(n)$. Then $K$ is a maximal compact subgroup of $G$. On $g = \text{Lie}(G)$ we put the inner product $\langle X, Y \rangle = \text{tr}(XY^*)$, Set $p = \text{Lie}(K)^\perp$ relative to this inner product.

We say that an element $\nu \in \mathbb{R}^n$ is $G$-critical if for any $X \in \text{Lie}(G)$, $\langle X\nu, \nu \rangle = 0$. The following is an extension of the Kempf-Ness Theorem first observed by Richardson and Slodoway.

**Theorem.** Let $G, K$ be as above. Let $\nu \in \mathbb{R}^n$.

1. If $\nu$ is critical if and only if $\|g\nu\| \geq \|\nu\|$ for all $g \in G$.
2. If $\nu$ is critical and if $w \in G\nu$ is such that $\|\nu\| = \|w\|$ then $w \in K\nu$.
3. If $G\nu$ is closed then there exists a critical element in $G\nu$.
4. If $\nu$ is critical then $G\nu$ is closed.
We set $V = \mathbb{R}^n$ as a $G$–module and $\text{Crit}_G(V)$ equal to the set of all critical vectors. If $X_1, \ldots, X_r$ is an orthonormal basis of $\mathfrak{p}$ then

$$\phi(v) = \sum \langle X_j v, v \rangle^2$$

is non-negative homogeneous polynomial of degree 4 defining $\text{Crit}_G(V)$. 


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We consider $\mathbb{R}^n$ as $n \times 1$ columns and thus if $v \in V$ then $v^*$ is $v$ as a row vector. So for $v, w \in V$, $vw^*$ is an $n \times n$ matrix and

$$\langle Xv, w \rangle = \text{tr}Xvw^*.$$
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$$\langle Xv, w \rangle = \text{tr} Xvw^*.$$ 

Let $P_g$ be the orthogonal projection of $M_n(\mathbb{R})$ onto $\mathfrak{g}$ then

$$\nabla \phi(v) = 4P_g(vv^*)v \in T_v(Gv).$$
We set $V = \mathbb{R}^n$ as a $G$–module and $\text{Crit}_G(V)$ equal to the set of all critical vectors. If $X_1, \ldots, X_r$ is an orthonormal basis of $\mathfrak{p}$ then

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$$\nabla \phi(v) = 4P_g(vv^*)v \in T_v(Gv).$$

Also note that $\nabla \phi(kv) = k\nabla \phi(v)$ for $k \in K$. 
Let $F(t, x)$ be the gradient flow corresponding to $\phi$. Then we have shown using freshman calculus that for $t > 0$ and $\|x\| \leq r$

$$\phi(F(t, x)) \leq \frac{C(r)}{t}.$$
Let \( F(t, x) \) be the gradient flow corresponding to \( \phi \). Then we have shown using freshman calculus that for \( t > 0 \) and \( \|x\| \leq r \)

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\phi(F(t, x)) \leq \frac{C(r)}{t}.
\]

In addition if \( Z \subset V \) is closed and \( G \)-invariant then \( F(t, Z) \subset Z \) and 2 in the real Kempf-Ness theorem implies:

\[
\lim_{t \to 1} L(t, K)\bigg|_{K = V} \text{ converges uniformly on compacta and this yields a strict deformation retraction of } Z/K \text{ to } (\text{Crit } G(V)) \setminus Z/K \text{ for any } G \text{-invariant closed subset of } V.
\]

The statement of the next result is simplified.

Corollary. If \( Z \) is Zariski closed in \( V \) then the GIT quotient, \( Z//G \), of \( Z \) is a strict deformation retract of \( Z/K \).

This is a very useful result since if \( G \) is connected \( K \) is connected and this implies that \( Z//G \) has path lifting. In the complex case this is an important result of Kraft, Petrie and Randall.

We now consider the result implied by using the deep results of Lojasiewicz.
Let $F(t, x)$ be the gradient flow corresponding to $\phi$. Then we have shown using freshman calculus that for $t > 0$ and $\|x\| \leq r$

$$\phi(F(t, x)) \leq \frac{C(r)}{t}.$$  

In addition if $Z \subseteq V$ is closed and $G$–invariant then $F(t, Z) \subseteq Z$ and 2 in the real Kempf-Ness theorem implies:

**Theorem.** Setting $L(t, K\nu) = KF(\frac{t}{1-t}, \nu)$ $0 \leq t < 1$ then $\lim_{t \to 1} L(t, K\nu)$ converges uniformly on compacta and this yields a strict deformation retraction of $Z/K$ to $(\text{Crit}_G(V) \cap Z)/K$ for any $G$–invariant closed subset of $V$. 

The statement of the next result is simplified.

**Corollary.** If $Z$ is Zariski closed in $V$ then the GIT quotient, $Z//G$, of $Z$ is a strict deformation retract of $Z/K$.

This is a very useful result since if $G$ is connected $K$ is connected and this implies that $Z//G$ has path lifting. In the complex case this is an important result of Kraft, Petrie and Randall.
Let $F(t, x)$ be the gradient flow corresponding to $\phi$. Then we have shown using freshman calculus that for $t > 0$ and $\|x\| \leq r$

$$\phi(F(t, x)) \leq \frac{C(r)}{t}.$$ 

In addition if $Z \subset V$ is closed and $G$–invariant then $F(t, Z) \subset Z$ and 2 in the real Kempf-Ness theorem implies:

**Theorem.** Setting $L(t, Kv) = KF(\frac{t}{1-t}, v)$ $0 \leq t < 1$ then $\lim_{t \to 1} L(t, Kv)$ converges uniformly on compacta and this yields a strict deformation retraction of $Z/K$ to $(Crit_G(V) \cap Z)/K$ for any $G$–invariant closed subset of $V$.

The statement of the next result is simplified.
Let $F(t, x)$ be the gradient flow corresponding to $\phi$. Then we have shown using freshman calculus that for $t > 0$ and $\|x\| \leq r$

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**Corollary.** If $Z$ is Zariski closed in $V$ then the GIT quotient, $Z//G$, of $Z$ is a strict deformation retract of $Z/K$. 

There are several additional results and applications that can be derived using this approach.
Let $F(t, x)$ be the gradient flow corresponding to $\phi$. Then we have shown using freshman calculus that for $t > 0$ and $\|x\| \leq r$

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**Corollary.** If $Z$ is Zariski closed in $V$ then the GIT quotient, $Z//G$, of $Z$ is a strict deformation retract of $Z/K$.

This is a very useful result since if $G$ is connected $K$ is connected and this implies that $Z//G$ has path lifting. In the complex case this is an important result of Kraft, Petrie and Randall.
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We now consider the result implied by using the deep results of Lojasiewicz.
The Lojasiewicz argument implies that if we set \( L(t, \nu) = F\left(\frac{t}{1-t}, \nu\right) \) then \( \lim_{t \to 1} H(t, \nu) \) converges uniformly on compacta.
The Lojasiewicz argument implies that if we set \( L(t, v) = F\left(\frac{t}{1-t}, v\right) \) then \( \lim_{t \to 1} H(t, v) \) converges uniformly on compacta.

**Theorem.** Let \( Z \subset V \) be closed and \( G \) invariant then \( L : [0, 1] \times Z \to Z \) defines a strong, \( K \)-equivariant deformation retraction of \( Z \) onto \( Z \cap \text{Crit}_G(V) \).
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Over \( \mathbb{C} \) this result is due to Neeman.
\[ C^n = V \oplus iV \] so as a real vector space we write it as \( V \oplus V = \mathbb{R}^{2n} \).

The real part of the standard Hermitian inner product on \( C^n \) becomes the standard inner product on \( \mathbb{R}^{2n} \). \( M_n(\mathbb{C}) \) becomes the algebra of \( 2 \times 2 \) block \( n \times n \) matrices

\[
\begin{bmatrix}
X & -Y \\
Y & X
\end{bmatrix}.
\]

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If \( X \subset \mathbb{C}^n \) is Zariski closed and defined by \( f_1, ..., f_k \) in \( \mathbb{C}[x_1, ..., x_n] \) then it is defined by \( \phi(x, y) = \sum |f_j(x + iy)|^2 \) as a real variety.
\( \mathbb{C}^n = V \oplus iV \) so as a real vector space we write it as \( V \oplus V = \mathbb{R}^{2n} \). The real part of the standard Hermitian inner product on \( \mathbb{C}^n \) becomes the standard inner product on \( \mathbb{R}^{2n} \). \( M_n(\mathbb{C}) \) becomes the algebra of \( 2 \times 2 \) block \( n \times n \) matrices

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If \( G \subset GL(n, \mathbb{C}) \) is a Zariski closed subgroup invariant under adjoint then \( G \) as a subgroup of \( GL(2n, \mathbb{R}) \) is invariant under transpose. Furthermore, if we define the critical set for the action of \( G \) on \( \mathbb{C}^n \) to be

\[
\{ v \in \mathbb{C}^n | \langle Xv, v \rangle = 0, X \in \text{Lie}(G) \}
\]

then this set is exactly \( \text{Crit}_G(\mathbb{R}^{2n}) \).
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The original Kempf-Ness theorem is now a special case of the real Kempf-Ness theorem since Zariski closure of complex orbits is the same as the closure in the metric topology of \( \mathbb{R}^{2n} \).
The system in the abstract for my talk is just the case of $GL(n, \mathbb{C})$ acting on $M_n(\mathbb{C})$ by conjugation. Yielding the gradient system

$$\dot{X} = -4[[X, X^*], X].$$
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Writing $F_\infty(X) = \lim_{t \to +\infty} F(t, X)$ then $F_\infty(X)$ is a normal operator with the same eigenvalues as $X$. 