Branching algebras for classical groups

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Survey on some of the works done by Roger Howe and his collaborators (Jackson, Kim, Lee, Tan, Wang, Willenbring) on branching algebras.
Setting:

$G$: complex classical group

$H$: certain subgroup of $G$ (mostly symmetric subgroup)

Examples of $(G, H)$: $(\text{GL}_n, \text{O}_n)$, $(\text{Sp}_{2n}, \text{GL}_n)$, $(\text{GL}_n \times \text{GL}_n, \text{GL}_n)$
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Branching problem for $(G, H)$
If $V$ be an irreducible rational $G$ module, what is $V|_H$?

(1) We have
\[ V|_H = \bigoplus_U m_{U, V} U \]
where the $U$s are irreducible $H$ modules.

Determine the branching multiplicities $m(U, V)$.

(2) Describe the $H$ submodules of $V$. 
Use highest weight theory:

Let \( B_H = A_H U_H \) be a Borel subgroup of \( H \), and consider

\[
V^{U_H} = \{ v : g.v = v \ \forall g \in U_H \}.
\]

This is a module for \( A_H \), and

\[
V^{U_H} = \bigoplus_{\lambda} (V^{U_H})_{\lambda}
\]

where

\[
(V^{U_H})_{\lambda} = \{ v \in V^{U_H} : a.v = \lambda(a)v \ \forall a \in A_H \}
\]

\((H \text{ highest weight vectors of weight } \lambda)\)

Then

\[
V|_H \simeq \bigoplus_{\lambda} (\dim(V^{U_H})_{\lambda}) U_{\lambda}
\]

where

\( U_{\lambda} = \text{irreducible } H \text{ module with highest weight } \lambda. \)
Branching rule $G \downarrow H$: \[ V|_H \cong \bigoplus_{\lambda} \dim(V^U H) \lambda U\lambda \]

Questions:
1. How to calculate $\dim(V^U H) \lambda$?
2. Can we describe a basis for $(V^U H) \lambda$?
Howe’s approach:

(i) Consider a “concrete” algebra $\mathcal{R}_G$ with an $G$ action such that $\mathcal{R}_G$ is decomposed as a multiplicity free sum of irreducible $G$ submodules as

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(ii) Consider the subalgebra of $U_H$ invariants:

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(iii) The structure of $\mathcal{A}_{(G,H)}$ encodes part of the branching rule from $G$ to $H$, so call it a branching algebra for $(G, H)$.

(iv) Study the branching algebra $\mathcal{A}_{(G,H)}$. 
Basic example:

\[ G = \text{GL}_n \times \text{GL}_n, \quad H = \Delta(\text{GL}_n) = \{(g, g) : g \in \text{GL}_n\}. \]
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Polynomial representations of \( \text{GL}_n \) are parametrized by Young diagrams with at most \( n \) rows (i.e. with depth \( \leq n \)).

\[ D \text{ (Young diagram)} \longrightarrow \rho^D_n \text{ (representation of } \text{GL}_n). \]
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\[ D \text{ (Young diagram)} \longrightarrow \rho_n^D \text{ (representation of } \text{GL}_n). \]

Example of a Young diagram:

\[
D = \begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
= (6,4,4,2) \text{ or } (6,4,4,2,0) \text{ etc}
\]
Branching problem for \((G, H) = (\text{GL}_n \times \text{GL}_n, \text{GL}_n)\):

For Young diagrams \(D\) and \(E\), \(\rho_n^D \otimes \rho_n^E\) is an irreducible module for \(\text{GL}_n \times \text{GL}_n\).

Restrict the action to \(\text{GL}_n = \Delta(\text{GL}_n)\), and describe the \(\text{GL}_n\) module structure of \(\rho_n^D \otimes \rho_n^E\).
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**In other words, we want to decompose the \(\text{GL}_n\) tensor product**

\[ \rho_n^D \otimes \rho_n^E. \]

So the branching rule in this case is **the Littlewood-Richardson (LR) Rule**:

\[ \rho_n^D \otimes \rho_n^E = \bigoplus_F c_{D,E}^F \rho_n^F, \]

where \(c_{D,E}^F\) is the number of LR tableaux of shape \(F/D\) and content \(E\).
We want to construct a branching algebra $\mathcal{A}_{(G,H)}$ which encodes the LR rule.
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Then

$$\mathcal{A}_{(G,H)} := \mathcal{R}_G^{U_H} \quad \text{where} \quad U_H = U_n = \left\{ \begin{pmatrix} 1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \in \text{GL}_n \right\}.$$
The construction of $\mathcal{R}_G$:

$\text{GL}_n \times \text{GL}_k$ acts on the algebra $\mathcal{P}(M_{nk})$ of polynomial functions on $M_{nk}(\mathbb{C})$:

$$\mathcal{P}(M_{nk}) \cong \bigoplus_D \rho_n^D \otimes \rho_k^D \quad (\text{GL}_n, \text{GL}_k \text{ duality})$$
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Extracting $U_k$ invariants:

$$\mathcal{P}(\text{M}_{nk})^{U_k} \cong \bigoplus_D \rho_n^D \otimes (\rho_k^D)^{U_k} \cong \bigoplus_D \rho_n^D.$$
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Take another copy:

$$\mathcal{P}(M_{n\ell})^{U_{\ell}} \cong \bigoplus_{E} \rho_n^E \otimes (\rho_{\ell}^E)^{U_{\ell}} \cong \bigoplus_{E} \rho_n^E.$$
Form the tensor product:

\[ \mathcal{R}_G := \mathcal{P}(M_{nk})^U_k \otimes \mathcal{P}(M_{n\ell})^U_\ell \cong \left( \bigoplus_D \rho_n^D \right) \otimes \left( \bigoplus_E \rho_n^E \right) \cong \bigoplus_{D,E} \rho_n^D \otimes \rho_n^E \]
Form the tensor product:

\[ R_G := \mathcal{P}(M_{nk})^{U_k} \otimes \mathcal{P}(M_{n\ell})^{U_\ell} \simeq \left( \bigoplus_{D} \rho_n^D \right) \otimes \left( \bigoplus_{E} \rho_n^E \right) \simeq \bigoplus_{D,E} \rho_n^D \otimes \rho_n^E \]

Extract the \( U_n = \Delta(U_n) \) invariants:

\[ \mathcal{A}_{(G,H)} := R_G^{UH} = \left( \mathcal{P}(M_{nk})^{U_k} \otimes \mathcal{P}(M_{n\ell})^{U_\ell} \right)^{U_n} \simeq \bigoplus_{D,E} \left( \rho_n^D \otimes \rho_n^E \right)^{U_n}. \]
Form the tensor product:

\[ R_G := P(M_{nk})^{U_k} \otimes P(M_{n\ell})^{U_\ell} \cong \bigoplus_D \rho_n^D \otimes \rho_n^E \cong \bigoplus_{D,E} \rho_n^D \otimes \rho_n^E \]

Extract the \( U_n = \Delta(U_n) \) invariants:

\[ \mathcal{A}_{(G,H)} := R_G^{U_H} = \left( P(M_{nk})^{U_k} \otimes P(M_{n\ell})^{U_\ell} \right)^{U_n} \cong \bigoplus_{D,E} \left( \rho_n^D \otimes \rho_n^E \right)^{U_n} \]

It can be further decomposed as

\[ \mathcal{A}_{(G,H)} \cong \bigoplus_{D,E} \left\{ \bigoplus_F \left( \rho_n^D \otimes \rho_n^E \right)^{U_n}_F \right\} = \bigoplus_{D,E,F} \mathcal{A}_{(G,H)}^{(D,E,F)} \]

where

\[ \mathcal{A}_{(G,H)}^{(D,E,F)} = \left( \rho_n^D \otimes \rho_n^E \right)^{U_n}_F = \text{highest weight vectors of weight } F \text{ in } \rho_n^D \otimes \rho_n^E \]

\[ \dim \mathcal{A}_{(G,H)}^{(D,E,F)} = \text{multiplicity of } \rho_n^F \text{ in } \rho_n^D \otimes \rho_n^E \]

Howe et al. call \( \mathcal{A}_{(G,H)} \) a \( \text{GL}_n \) tensor product algebra.
It turns out that $\mathcal{A}(G,H)$ also encodes another branching rule:

$$\mathcal{A}(G,H) = R^U_H = \left( \mathcal{P}(M_{nk})^U_k \otimes \mathcal{P}(M_{n\ell})^U_\ell \right)^U_n \simeq \mathcal{P}(M_{nk} \oplus M_{n\ell})^U_{U_n \times U_k \times U_\ell}$$

$$\simeq \mathcal{P}(M_{n(k+\ell)})^U_{U_n \times U_k \times U_\ell} \simeq \left( \bigoplus_F \rho_n^F \otimes \rho_{k+\ell}^F \right)^U_{U_n \times U_k \times U_\ell}$$

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$\mathcal{A}_{(G,H)}$ encodes the branching rule for $\text{GL}_{k+\ell} \downarrow \text{GL}_k \times \text{GL}_\ell$.

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From this, we obtain the reciprocity law:

$$
\dim \mathcal{A}^{(D,E,F)}_{(G,H)} = \text{multiplicity of } \rho_k^D \otimes \rho_\ell^E \text{ in } \rho_n^F = \text{multiplicity of } \rho_n^F \text{ in } \rho_n^D \otimes \rho_n^E
$$
Problem: Find a basis for $\mathcal{A}_{(G,H)}$.

Since $\mathcal{A}_{(G,H)} = \bigoplus_{D,E,F} \mathcal{A}_{(G,H)}^{(D,E,F)}$, it suffices to find a basis for each subspace $\mathcal{A}_{(G,H)}^{(D,E,F)}$. 


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By the Littlewood-Richardson Rule,

\[
\dim \mathcal{A}_{(G,H)}^{(D,E,F)} = c_{D,E}^F
\]

\[
= \text{number of LR tableaux } T \text{ of shape } F/D \text{ and content } E.
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Plan: LR tableau $T \rightarrow$ construct a basis vector $\Delta_T$ in $\mathcal{A}_{(G,H)}^{(D,E,F)}$.
Now

\[ \mathcal{A}_{(G,H)} = \left( \mathcal{P}(M_{nk})^{U_k} \otimes \mathcal{P}(M_{n\ell})^{U_{\ell}} \right)^{U_n} \]

\[ = \mathcal{P}(M_{n,k} \oplus M_{n,\ell})^{U_n \times U_k \times U_{\ell}}, \]

it is a subalgebra of \( \mathcal{P}(M_{n,k} \oplus M_{n,\ell}) \).
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Write the coordinates of \( M_{n,k} \oplus M_{n,\ell} \) as

\[
\begin{pmatrix}
  x_{11} & x_{12} & \cdots & x_{1k} & y_{11} & y_{12} & \cdots & y_{1\ell} \\
  x_{21} & x_{22} & \cdots & x_{2k} & y_{21} & y_{22} & \cdots & y_{2\ell} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  x_{n1} & x_{n2} & \cdots & x_{nk} & y_{n1} & y_{n2} & \cdots & y_{n\ell}
\end{pmatrix}
\]

Then each \( \Delta_T \) is a polynomial on these variables.
Associate each skew tableau $T$ with a monomial $m_T$.

**Example:** $T = \begin{array}{ccc}
1 \\
1 \\
2
\end{array} \rightarrow \begin{array}{ccc}
x_{11} & x_{11} & y_{11} \\
x_{22} & y_{21} \\
y_{32}
\end{array} \rightarrow m_T = (x_{11}x_{22}y_{11}y_{32})(x_{11}y_{21})$
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Introduce a monomial ordering: the graded lexicographic order with

$$x_{11} > x_{21} > \cdots > x_{n1} > x_{12} > \cdots > x_{nk} > y_{11} > y_{21} > \cdots > y_{n\ell}.$$

$$\text{LM}(f) = \text{leading monomial of } f.$$
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\hline
\hline
\hline
\hline
\hline
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\hline
\hline
\hline
\hline
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**Theorem (Howe-Tan-Willenbring, Advances 2005)**

$\mathcal{A}^{(D,E,F)}_{(G,H)}$ has a basis $\{\Delta_T\}$ with the property that for each $T$,

$\text{LM}(\Delta_T) = m_T$. 

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Example. Let $D = \begin{array}{c} \end{array}$, $E = \begin{array}{c} \end{array}$, $F = \begin{array}{c} \end{array}$.

Then $\rho_n^F$ occurs in $\rho_n^D \otimes \rho_n^E$ with multiplicity 2.
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1 & 2 \\
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1 & 1 \\
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1 & 2 \\
\end{array}$.

Then $\rho_n^F$ occurs in $\rho_n^D \otimes \rho_n^E$ with multiplicity 2.

$$T_1 = \begin{array}{ccc}
1 & 1 \\
1 & 2 \\
\end{array} \quad \Delta_{T_1} = \begin{vmatrix}
x_{11} & x_{12} & y_{11} & y_{12} \\
x_{21} & x_{22} & y_{21} & y_{22} \\
x_{31} & x_{32} & y_{31} & y_{32} \\
0 & 0 & y_{11} & y_{12} \\
\end{vmatrix} \begin{vmatrix}
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\end{vmatrix}$$

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$$T_2 = \begin{array}{ccc}
1 & 2 & 1 \\
1 & & \\
\end{array}$$ 

$$\Delta_{T_2} = \begin{vmatrix}
x_{11} & x_{12} & y_{11} \\
x_{21} & x_{22} & y_{21} \\
x_{31} & x_{32} & y_{31} \\
0 & 0 & y_{11} & y_{12}
\end{vmatrix}$$

$$\text{LM}(\Delta_{T_2}) = (x_{11}x_{22}y_3)(x_{11}y_{11}y_{22}) = m_{T_2}$$
Let
\[ S_{(G,H)} = \{ \text{LM}(f) : f \in \mathcal{A}_{(G,H)}, f \neq 0 \} = \{ m_T \}. \]
Then \( S_{(G,H)} \) is a semigroup because \( \mathcal{A}_{(G,H)} \) is an algebra and
\[ \text{LM}(f_1 f_2) = \text{LM}(f_1) \text{LM}(f_2). \]
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**What we can we say about this semigroup \( S_{(G,H)} \)?**
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**What we can we say about this semigroup \( S_{(G,H)} \)?**

There is a rational polyhedral cone \( C \) in some \( \mathbb{R}^N \) such that 

\[ S_{(G,H)} \cong C \cap \mathbb{Z}^N. \]

It is finitely generated.
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The polyhedral cone \( C \) is called the **Littlewood-Richardson cone**
by Igor Pak, and
\[ c_{D,E}^F = \text{number of integral points in a polytope contained in } C. \]
The **initial algebra** in($\mathcal{A}_{(G,H)}$) of $\mathcal{A}_{(G,H)}$ is the subalgebra of $\mathcal{P}(M_{nk} \oplus M_{nl})$ generated by $S_{(G,H)}$. 
The **initial algebra** \( \text{in}(\mathcal{A}_{(G,H)}) \) of \( \mathcal{A}_{(G,H)} \) is the subalgebra of \( \mathcal{P}(M_{nk} \oplus M_{nl}) \) generated by \( S_{(G,H)} \).

So

\[
\text{in}(\mathcal{A}_{G,H}) \cong \mathbb{C}[S_{(G,H)}]
\]

is the **semigroup algebra** on \( S_{(G,H)} \), and it is finitely generated.
The **initial algebra** \( \text{in}(\mathcal{A}_{(G,H)}) \) of \( \mathcal{A}_{(G,H)} \) is the subalgebra of \( \mathcal{P}(M_{nk} \oplus M_{nl}) \) generated by \( S_{(G,H)} \).

So

\[
\text{in}(\mathcal{A}_{G,H}) \simeq \mathbb{C}[S_{(G,H)}]
\]

is the **semigroup algebra** on \( S_{(G,H)} \), and it is finitely generated.

By a general results of Conca, Herzog, and Valla, we have:

**Theorem ([HJLTW]).** *The semigroup algebra \( \mathbb{C}[S_{(G,H)}] \) is a flat deformation of \( \mathcal{A}_{(G,H)} \).*
Similar results also hold for the following symmetric pairs (under a stable range condition):

\[(\text{GL}_n, \text{O}_n), \ (\text{O}_{n+m}, \text{O}_n \times \text{O}_m), \ (\text{Sp}_{2n}, \text{GL}_n), \ (\text{GL}_{2n}, \text{Sp}_{2n}), \ (\text{Sp}_{2(n+m)}, \text{Sp}_{2n} \times \text{Sp}_m), \ (\text{O}_{2n}, \text{GL}_n)\]

Branching multiplicities in these cases can be deduced from the algebra structure and the LR rule.
**m-fold tensor product algebra**

This is a branching algebra $\mathcal{A}_{(G,H)}$ which describes the decomposition of $m$-fold tensor products of $\text{GL}_n$ modules:

$$\rho^D_1 \otimes \rho^D_2 \otimes \cdots \otimes \rho^D_m$$

where

$$G = \text{GL}_n^m, \quad H = \Delta(\text{GL}_n).$$
**m-fold tensor product algebra**

This is a branching algebra \( \mathcal{A}_{(G,H)} \) which describes the decomposition of \( m \)-fold tensor products of \( \text{GL}_n \) modules:

\[
\rho_n^{D_1} \otimes \rho_n^{D_2} \otimes \cdots \otimes \rho_n^{D_m}
\]

where

\[
G = \text{GL}_n^m, \quad H = \Delta(\text{GL}_n).
\]

**A Special case:** tensor product of the form

\[
\rho_n^{D} \otimes \rho_n^{(\alpha_1)} \otimes \rho_n^{(\alpha_2)} \otimes \cdots \otimes \rho_n^{(\alpha_\ell)} \simeq \rho_n^{D} \otimes S^{\alpha_1}(\mathbb{C}^n) \otimes S^{\alpha_2}(\mathbb{C}^n) \otimes \cdots \otimes S^{\alpha_\ell}(\mathbb{C}^n).
\]

We call a description of this tensor product **an iterated Pieri rule**.
An algebra which encodes the iterated Pieri rule:

$$\mathcal{P}(M_{n(k+\ell)}) = \mathcal{P}(M_{nk} \oplus \mathbb{C}^n \oplus \mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n)$$

$$= \mathcal{P}(M_{nk}) \otimes \mathcal{P}(\mathbb{C}^n) \otimes \mathcal{P}(\mathbb{C}^n) \otimes \cdots \otimes \mathcal{P}(\mathbb{C}^n)$$

$$\simeq \bigoplus_{D} \rho^D_n \otimes \rho^D_k \otimes \bigoplus_{\alpha_1} \rho^{(\alpha_1)}_n \otimes \cdots \otimes \bigoplus_{\alpha_\ell} \rho^{(\alpha_\ell)}_n$$
An algebra which encodes the iterated Pieri rule:

\[ \mathcal{P}(M_{n(k+\ell)}) = \mathcal{P}(M_{nk} \oplus \mathbb{C}^n \oplus \mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n) \]

\[ = \mathcal{P}(M_{nk}) \otimes \mathcal{P}(\mathbb{C}^n) \otimes \mathcal{P}(\mathbb{C}^n) \otimes \cdots \otimes \mathcal{P}(\mathbb{C}^n) \]

\[ \simeq \bigoplus_{D} \rho_D^{D_n} \otimes \rho_k^{D_k} \otimes \bigoplus_{\alpha_1} \rho_n^{(\alpha_1)} \otimes \bigoplus_{\alpha_\ell} \rho_n^{(\alpha_\ell)} \]

Extract \( U_n \times U_k \) invariants:

\[ \mathcal{P}(M_{n(k+\ell)})^{U_n \times U_k} \simeq \bigoplus_{D, \alpha} \left( \rho_D^{D_n} \otimes \rho_n^{(\alpha_1)} \otimes \rho_n^{(\alpha_2)} \otimes \cdots \otimes \rho_n^{(\alpha_\ell)} \right)^{U_n} \otimes \rho_k^{D_k} \]

We call \( \mathcal{P}(M_{n(k+\ell)})^{U_n \times U_k} \) an *iterated Pieri algebra* for GL\(_n\).
The iterated Pieri algebra $\mathcal{P}(M_{n(k+\ell)})^{U_n \times U_k}$ also encodes the branching rule for

$$GL_{k+\ell} \downarrow GL_k \times GL_1^\ell = GL_k \times (GL_1 \times \cdots \times GL_1).$$
The iterated Pieri algebra $\mathcal{P}(M_{n(k+\ell)})^{U_n \times U_k}$ also encodes the branching rule for

$$GL_{k+\ell} \downarrow GL_k \times GL_1^\ell = GL_k \times (GL_1 \times \cdots \times GL_1).$$

**Special case:** If $k = 1$, then this is branching for

$$GL_{\ell+1} \downarrow = GL_1^{\ell+1} = \underbrace{GL_1 \times \cdots \times GL_1}_{\ell+1}.$$ 

That is, decompose $\rho_{\ell+1}^D$ into weight spaces, and find a basis of each weight space.
Comparing tensor product algebra with iterated Pieri algebra

$GL_n$ tensor product algebra:

$\mathcal{P}(M_{n(k+\ell)})^{U_n \times U_k \times U_\ell}$ describes general tensor products $\rho_n^D \otimes \rho_n^E$. 
Comparing tensor product algebra with iterated Pieri algebra

**GL\(_n\) tensor product algebra:**

\[ \mathcal{P}(M_{n(k+\ell)})^{U_n \times U_k \times U_\ell} \] describes general tensor products \( \rho_n^D \otimes \rho_n^E \).

**Iterated Pieri algebra for GL\(_n\):**

\[ \mathcal{P}(M_{n(k+\ell)})^{U_n \times U_k} \] describes tensor products of the form

\[ \rho_n^D \otimes \rho_n^{(\alpha_1)} \otimes \rho_n^{(\alpha_2)} \otimes \cdots \otimes \rho_n^{(\alpha_\ell)}. \]
Comparing tensor product algebra with iterated Pieri algebra

\textbf{GL}_n \text{ tensor product algebra:}

\[ \mathcal{P}(M_{n(k+\ell)})^{U_n \times U_k \times U_\ell} \] describes general tensor products \[ \rho_n^D \otimes \rho_n^E. \]

\textbf{Iterated Pieri algebra for } \textbf{GL}_n : \quad \text{for } n \geq k \geq \ell \geq 0

\[ \mathcal{P}(M_{n(k+\ell)})^{U_n \times U_k} \] describes tensor products of the form

\[ \rho_n^D \otimes \rho_n^{(\alpha_1)} \otimes \rho_n^{(\alpha_2)} \otimes \cdots \otimes \rho_n^{(\alpha_\ell)}. \]

We have

\[ \mathcal{P}(M_{n(k+\ell)})^{U_n \times U_k \times U_\ell} \subseteq \mathcal{P}(M_{n(k+\ell)})^{U_n \times U_k} \]

By analyzing how the tensor product algebra sits inside the iterated Pieri algebra, we can give a proof of the Littlewood-Richardson Rule ([Howe-Lee], BAMS 2012).
What is the semigroup $S$ associated with the iterated Pieri algebra $\mathcal{P}(M_{n(k+\ell)})^{U_n \times U_k}$?

The elements of $S$ should count the multiplicity in the tensor product

$$\rho_n^D \otimes \rho_n^{(\alpha_1)} \otimes \rho_n^{(\alpha_2)} \otimes \cdots \otimes \rho_n^{(\alpha_\ell)}.$$
What is the semigroup $S$ associated with the iterated Pieri algebra $P(M_{n(k+\ell)})^{U_n \times U_k}$?

The elements of $S$ should count the multiplicity in the tensor product

$$\rho_n^D \otimes \rho_n^{(\alpha_1)} \otimes \rho_n^{(\alpha_2)} \otimes \cdots \otimes \rho_n^{(\alpha_\ell)}.$$

By the Pieri Rule,

$$\rho_p^D \otimes \rho_p^{(\alpha_1)} = \bigoplus_F \rho_p^F$$

(multiplicity free)

where $F$ satisfies the interlacing condition: If $D = (d_1, \ldots, d_p)$ and $F = (f_1, \ldots, f_p)$, then

$$f_1 \geq d_1 \geq f_2 \geq d_2 \geq \cdots \geq f_p \geq d_p.$$
What is the semigroup $S$ associated with the iterated Pieri algebra $\mathcal{P}(M_{n(k+\ell)})^{U_n \times U_k}$?

The elements of $S$ should count the multiplicity in the tensor product

$$
\rho_n^{D} \otimes \rho_n^{(\alpha_1)} \otimes \rho_n^{(\alpha_2)} \otimes \cdots \otimes \rho_n^{(\alpha_{\ell})}.
$$

By the Pieri Rule,

$$
\rho_p^{D} \otimes \rho_p^{(\alpha_1)} = \bigoplus_F \rho_p^F \quad \text{ (multiplicity free)}
$$

where $F$ satisfies the interlacing condition: If $D = (d_1, \ldots, d_p)$ and $F = (f_1, \ldots, f_p)$, then

$$
f_1 \geq d_1 \geq f_2 \geq d_2 \geq \cdots \geq f_p \geq d_p.
$$

We indicate these inequalities by writing

$$
d_1 \quad d_2 \quad \cdots \quad d_p
$$

$$
f_1 \quad f_2 \quad \cdots \quad f_p
$$
By iterating the Pieri Rule,
\[ \rho_n^D \otimes \rho_n^{(\alpha_1)} \otimes \rho_n^{(\alpha_2)} \otimes \cdots \otimes \rho_n^{(\alpha_\ell)} = \bigoplus_F m_F \rho_n^F \]

where \( m_F \) is the number of “Gelfand-Zeltlin” of the form

\[
\lambda =
\begin{pmatrix}
\lambda_{10} & \lambda_{20} & \cdots & \lambda_{n0} \\
\lambda_{11} & \lambda_{21} & \cdots & \lambda_{n1} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{1\ell} & \lambda_{2\ell} & \cdots & \lambda_{n\ell}
\end{pmatrix}
\]

where \( D = (\lambda_{10}, \lambda_{20}, \cdots, \lambda_{p0}) \) and \( F = (\lambda_{1\ell}, \lambda_{2\ell}, \cdots, \lambda_{n\ell}) \).
By iterating the Pieri Rule,
\[ \rho_n^D \otimes \rho_n^{(\alpha_1)} \otimes \rho_n^{(\alpha_2)} \otimes \cdots \otimes \rho_n^{(\alpha_\ell)} = \bigoplus_{F} m_F \rho_n^F \]
where \( m_F \) is the number of “Gelfand-Zeltlin” of the form
\[ \lambda = \begin{array}{cccc}
\lambda_{10} & \lambda_{20} & \cdots & \lambda_{n0} \\
\lambda_{11} & \lambda_{21} & \cdots & \lambda_{n1} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{1\ell} & \lambda_{2\ell} & \cdots & \lambda_{n\ell}
\end{array} \]
where \( D = (\lambda_{10}, \lambda_{20}, \cdots, \lambda_{p0}) \) and \( F = (\lambda_{1\ell}, \lambda_{2\ell}, \cdots, \lambda_{n\ell}) \).

These patterns can be viewed as **order preserving functions on a poset** \( \Gamma \)
\[ \lambda : \Gamma \to \mathbb{Z}^+ . \]
The set

\[(\mathbb{Z}^+)_{\Gamma; \geq} = \{ f : \Gamma \to \mathbb{Z}^+ | f \text{ is order preserving} \}\]

forms a semigroup, and is called a \textbf{Hibi cone}. It has a very simple semigroup structure.

(More generally, we can replace \(\Gamma\) by a finite poset)
The set

$$(\mathbb{Z}^+)_{\Gamma, \geq} = \{f : \Gamma \to \mathbb{Z}^+ | f \text{ is order preserving}\}$$

forms a semigroup, and is called a **Hibi cone**. It has a very simple semigroup structure.

(More generally, we can replace $\Gamma$ by a finite poset)

Call a subset $A$ of $\Gamma$ **increasing** if

$$a \in A, \ x \in \Gamma, \ x \geq a \implies x \in A.$$ 

Denote by $J^*(\Gamma)$ the collection of all increasing subsets of $\Gamma$. 

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For each $A \in J^*(\Gamma)$, let

$$\chi_A(x) = \begin{cases} 
1 & x \in A \\
0 & x \notin A.
\end{cases}$$

Then clearly $\chi_A \in (\mathbb{Z}^+)\overline{\Gamma},\geq$. 
For each $A \in J^*(\Gamma)$, let

$$\chi_A(x) = \begin{cases} 1 & x \in A \\
0 & x \notin A. \end{cases}$$

Then clearly $\chi_A \in (\mathbb{Z}^+)^{\Gamma,\succeq}$.

**Theorem.** The semigroup $(\mathbb{Z}^+)^{\Gamma,\succeq}$ is generated by $\{\chi_A : A \in J^*(\Gamma)\}$ and it has relations

$$\chi_A + \chi_B = \chi_{A \cup B} + \chi_{A \cap B}, \quad A, B \in J^*(\Gamma).$$
For each $A \in J^*(\Gamma)$, let

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

Then clearly $\chi_A \in \Omega_{\Gamma}$.

**Theorem.** The semigroup $(\mathbb{Z}^+)_{\geq}^\Gamma$ is generated by $\{\chi_A : A \in J^*(\Gamma)\}$ and it has relations

$$\chi_A + \chi_B = \chi_{A \cup B} + \chi_{A \cap B}, \quad A, B \in J^*(\Gamma).$$

It follows that every $f \in (\mathbb{Z}^+)_{\geq}^\Gamma$ can be expressed as

$$f = \sum_j c_j \chi_{A_j}$$

where $c_j \in \mathbb{N}$ and $A_1 \subset A_2 \subset \cdots \subset A_N = \Gamma$ is a chain in $J^*(\Gamma)$. 65
In the case when \( n = 3, k = \ell = 2 \), \((\mathbb{Z}^+)_{\Gamma, \geq}\) consists of patterns of the form

\[
\lambda = \begin{pmatrix}
\lambda_{10} & \lambda_{20} & 0 \\
\lambda_{11} & \lambda_{21} & \lambda_{31} \\
\lambda_{12} & \lambda_{22} & \lambda_{32}
\end{pmatrix}
\]
In the case when $n = 3, k = \ell = 2$, $(\mathbb{Z}^+)^{\Gamma, \geq}$ consists of patterns of the form

\[
\lambda = \begin{pmatrix}
\lambda_{10} & \lambda_{20} & 0 \\
\lambda_{11} & \lambda_{21} & \lambda_{31} \\
\lambda_{12} & \lambda_{22} & \lambda_{32}
\end{pmatrix}
\]

The generators $\chi_A$ of $(\mathbb{Z}^+)^{\Gamma, \geq}$ are:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]
For general $n, k, \ell$, each generator $\chi_A$ of $(\mathbb{Z}^+)^{\Gamma_{\geq}}$ corresponds to an element in $\mathcal{P}(M_{n(k+\ell)})^{U_n \times U_k}$ of the form

$$\delta_A = \begin{vmatrix}
    x_{11} & x_{12} & \cdots & x_{1p} & y_{1s_1} & y_{1s_2} & \cdots & y_{1s_q} \\
    x_{21} & x_{22} & \cdots & x_{2p} & y_{2s_1} & y_{2s_2} & \cdots & y_{2s_q} \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    x_{(p+q)1} & x_{(p+q)2} & \cdots & x_{(p+q)p} & y_{(p+q)s_1} & y_{(p+q)s_2} & \cdots & y_{(p+q)s_q}
\end{vmatrix}.$$ 

Let $Q$ be the set of all $\delta_A$. 

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For general $n, k, \ell$, each generator $\chi_A$ of $(\mathbb{Z}^+)^{\Gamma_\geq}$ corresponds to an element in $\mathcal{P}(M_{n(k+\ell)})^{U_n \times U_k}$ of the form

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\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
x_{(p+q)1} & x_{(p+q)2} & \cdots & x_{(p+q)p} & y_{(p+q)s_1} & y_{(p+q)s_2} & \cdots & y_{(p+q)s_q}
\end{vmatrix}.$$ 

Let $Q$ be the set of all $\delta_A$.

If $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_r$, then we call the product

$$\delta_{A_1} \delta_{A_2} \cdots \delta_{A_r}$$

a *standard monomial* on $Q$. 

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For general \( n, k, \ell \), each generator \( \chi_A \) of \( (\mathbb{Z}^+)_{\Gamma; \geq} \) corresponds to an element in \( P(M_{n(k+\ell)})^{U_n \times U_k} \) of the form

\[
\delta_A = \begin{vmatrix}
    x_{11} & x_{12} & \cdots & x_{1p} & y_{1s_1} & y_{1s_2} & \cdots & y_{1s_q} \\
    x_{21} & x_{22} & \cdots & x_{2p} & y_{2s_1} & y_{2s_2} & \cdots & y_{2s_q} \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    x_{(p+q)1} & x_{(p+q)2} & \cdots & x_{(p+q)p} & y_{(p+q)s_1} & y_{(p+q)s_2} & \cdots & y_{(p+q)s_q}
\end{vmatrix}.
\]

Let \( Q \) be the set of all \( \delta_A \).

If \( A_1 \subseteq A_2 \subseteq \cdots \subseteq A_r \), then we call the product

\[
\delta_{A_1} \delta_{A_2} \cdots \delta_{A_r}
\]

a **standard monomial** on \( Q \).

It turns out that the set of all standard monomials on \( Q \) forms a vector space basis for \( P(M_{n(k+\ell)})^{U_n \times U_k} \). We say that \( P(M_{n(k+\ell)})^{U_n \times U_k} \) has a standard monomial theory for \( Q \).

This treatment was given by **Sangjib Kim** in his thesis.
What other branching algebras are associated with Hibi cones?

The double Pieri algebra $\mathcal{L}_{(n,p),(k,q)}$ for $\text{GL}_n \times \text{GL}_k$

It describes

$$\left\{ \rho_n^D \otimes \left( \bigotimes_{i=1}^p \rho_n^{(\alpha_i)} \right) \right\} \otimes \left\{ \rho_k^D \otimes \left( \bigotimes_{j=1}^q \rho_k^{(\alpha_j)} \right) \right\}$$

with depth($D$) $\leq k \leq n$. 
What other branching algebras are associated with Hibi cones?

The double Pieri algebra $\mathcal{L}_{(n,p),(k,q)}$ for $\text{GL}_n \times \text{GL}_k$

It describes

$$\left\{ \rho_D \otimes \left( \bigotimes_{i=1}^{p} \rho_n^{(\alpha_i)} \right) \right\} \otimes \left\{ \rho_D \otimes \left( \bigotimes_{j=1}^{q} \rho_k^{(\alpha_j)} \right) \right\}$$

with $\text{depth}(D) \leq k \leq n$.

The iterated Pieri algebra $\mathcal{A}_{n,k,p}$ for $\text{O}_n$ where $2(k + p) < n$.

It describes

$$\sigma_D \otimes \left( \bigotimes_{i=1}^{\ell} \sigma_n^{(\alpha_i)} \right)$$

where $\sigma_D$ is the irreducible representation of $\text{O}_n$ labelled by $D$ and $\text{depth}(D) \leq k$. 

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The iterated Pieri algebra $Q_{n,k,p}$ for $\text{Sp}_{2n}$ where $k + p < n$.

It describes

$$\tau_{2n}^{D} \otimes \left( \bigotimes_{i=1}^{\ell} \tau_{2n}^{(\alpha_i)} \right)$$

where $\tau_{2n}^{D}$ is the irreducible representation of $\text{Sp}_{2n}$ labelled by $D$ and $\text{depth}(D) \leq k$. 
The iterated Pieri algebra $Q_{n,k,p}$ for $\text{Sp}_{2n}$ where $k + p < n$. It describes

$$
\tau_{2n}^D \otimes \left( \bigotimes_{i=1}^\ell \tau_{2n}^{(\alpha_i)} \right)
$$

where $\tau_{2n}^D$ is the irreducible representation of $\text{Sp}_{2n}$ labelled by $D$ and $\text{depth}(D) \leq k$.

It turns out that $Q_{n,k,p} \simeq \mathcal{A}_{2n,k,p}$ for $k + p < n$. 
The (more general) iterated Pieri algebra $\mathcal{A}_{n,k,\ell,p,q}$ for $\text{GL}_n$ where $k + p + \ell + q) \leq n$.

It describes

$$\rho_n^{D,E} \otimes \left( \bigotimes_{i=1}^{p} \rho_n^{(\alpha_i)} \right) \otimes \left( \bigotimes_{j=1}^{q} \rho_n^{(\alpha_i)^*} \right)$$

where $\text{depth}(D) \leq k$ and $\text{depth}(E) \leq \ell$. 
The (more general) iterated Pieri algebra $\mathcal{A}_{n,k,\ell,p,q}$ for $\text{GL}_n$ where $k + p + \ell + q \leq n$.

It describes

$$\rho_{D,E} \otimes \left( \bigotimes_{i=1}^{p} \rho_{n}(\alpha_i) \right) \otimes \left( \bigotimes_{j=1}^{q} \rho_{n}(\alpha_i)^* \right)$$

where $\text{depth}(D) \leq k$ and $\text{depth}(E) \leq \ell$.

It turns out that double Pieri algebras can be regarded as a common structure shared by the iterated Pieri algebras.

**Theorem.** We have the isomorphism of graded algebras

$$\mathcal{A}_{n,k,p} \cong \mathcal{L}(n,p),(k,p) \otimes \mathcal{P}(\wedge^2(\mathbb{C}^p)),$$

$$\mathcal{A}_{n,k,\ell,p,q} \cong \mathcal{L}(n,p),(k,q) \otimes \mathcal{L}(n,q),(\ell,p) \otimes \mathcal{P}(\mathbb{M}_{pq}).$$
Can the stable range condition be removed?
Can the stable range condition be removed?

**Antirow Pieri algebra for GLₙ** (without stable range condition)

\[ \mathcal{R}_{n,p,q} := \mathcal{P}(M_{np}) \otimes \left( \bigotimes_{i=1}^{q} \mathcal{P}(\mathbb{C}^{n*}_{i}) \right) \cong \left( \bigoplus_{D} \rho_{n}^{D} \otimes \rho_{p}^{D} \right) \otimes \left( \bigotimes_{i=1}^{q} \rho_{n}^{(\beta_{i})*} \right) \]

\[ \cong \bigoplus_{F,\alpha} \left\{ \rho_{n}^{D} \otimes \left( \bigotimes_{i=1}^{q} \rho_{n}^{(\beta_{i})*} \right) \right\} \otimes \rho_{p}^{F}. \]
Can the stable range condition be removed?

Antirow Pieri algebra for $GL_n$ (without stable range condition)

\[
R_{n,p,q} := \mathcal{P}(M_{np}) \otimes \left( \bigotimes_{i=1}^{q} \mathcal{P}(C_{i}^{n*}) \right) \simeq \left( \bigoplus_{D} \rho_{n}^{D} \otimes \rho_{p}^{D} \right) \otimes \left( \bigotimes_{i=1}^{q} \rho_{n}^{(\beta_{i})*} \right) \\
\simeq \bigoplus_{F,\alpha} \left\{ \rho_{n}^{D} \otimes \left( \bigotimes_{i=1}^{q} \rho_{n}^{(\beta_{i})*} \right) \right\} \otimes \rho_{p}^{F}. 
\]

Extract $GL_n \times GL_p$ highest weight vectors:

\[
R_{U_{n},U_{p}}^{U_{n} \times U_{p}} \simeq \bigoplus_{F,\alpha} \left\{ \rho_{n}^{D} \otimes \left( \bigotimes_{i=1}^{q} \rho_{n}^{(\beta_{i})*} \right) \right\} U_{n} \otimes \left( \rho_{p}^{F} \right) U_{p}. 
\]
Can the stable range condition be removed?

**Antirow Pieri algebra for GL\(_n\) (without stable range condition)**

\[
\mathcal{R}_{n,p,q} := \mathcal{P}(M_{np}) \otimes \left( \bigotimes_{i=1}^{q} \mathcal{P}(\mathbb{C}^{n^*}) \right) \simeq \left( \bigoplus_{D} \rho_n^D \otimes \rho_p^D \right) \otimes \left( \bigotimes_{i=1}^{q} \rho_n^{(\beta_i)^*} \right) \\
\simeq \bigoplus_{F,\alpha} \left\{ \rho_n^D \otimes \left( \bigotimes_{i=1}^{q} \rho_n^{(\beta_i)^*} \right) \right\} \otimes \rho_p^F.
\]

Extract \(\text{GL}_n \times \text{GL}_p \times A_q\) highest weight vectors:

\[
\mathcal{R}_{n,p,q}^{U_n \times U_p} \simeq \bigoplus_{F,\alpha} \left\{ \rho_n^D \otimes \left( \bigotimes_{i=1}^{q} \rho_n^{(\beta_i)^*} \right) \right\} \otimes (\rho_p^F)^{U_p}. \\
\]

So the algebra \(\mathcal{R}_{n,p,q}^{U_n \times U_p}\) describes \(\rho_n^D \otimes \left( \bigotimes_{i=1}^{q} \rho_n^{(\beta_i)^*} \right)\).
Multiplicities in $\rho_n^D \otimes \left( \bigotimes_{i=1}^{q} \rho_n^{(\beta_i)^*} \right)$ are counted by patterns of the form

$$\nu = \begin{array}{cccc}
\nu_{10} & \nu_{20} & \cdots & \nu_{n0} \\
\nu_{11} & \nu_{21} & \cdots & \nu_{n1} \\
\nu & \cdots & \cdots & \cdots \\
\nu_{1q} & \nu_{2q} & \cdots & \nu_{nq}
\end{array}$$

with $D = (\nu_{10}, \nu_{20}, \cdots, \nu_{n0})$. 
Multiplicities in $\rho_n^D \otimes \left( \bigotimes_{i=1}^{q} \rho_n^{(\beta_i)^*} \right)$ are counted by patterns of the form

$$\nu = \begin{bmatrix} \nu_{10} & \nu_{20} & \cdots & \nu_{n0} \\ \nu_{11} & \nu_{21} & \cdots & \nu_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{1q} & \nu_{2q} & \cdots & \nu_{nq} \end{bmatrix}$$

with $D = (\nu_{10}, \nu_{20}, \cdots, \nu_{n0})$.

Some of the entries $\nu_{ij}$ can be negative. The associated semigroup can be identified with a set of order preserving functions $f : \Gamma \to \mathbb{Z}$, and is called a signed Hibi cone.
Multiplicities in $\rho_n^D \otimes \left( \bigotimes_{i=1}^{q} \rho_n^{(\beta_i)^*} \right)$ are counted by patterns of the form

$$\nu = \begin{array}{cccc}
\nu_{10} & \nu_{20} & \cdots & \nu_{n0} \\
\nu_{11} & \nu_{21} & & \nu_{n1} \\
& & \ddots & \\
& & & \nu_{1q} & \nu_{2q} & \cdots & \nu_{nq}
\end{array}$$

with $D = (\nu_{10}, \nu_{20}, \cdots, \nu_{n0})$.

Some of the entries $\nu_{ij}$ can be negative. The associated semigroup can be identified with a set of order preserving functions $f : \Gamma \to \mathbb{Z}$, and is called a **signed Hibi cone**.

The structure of the signed Hibi cone and the algebra were determined in Yi Wang’s thesis (2013).
Thank you.