ON LANGLANDS' AUTOMORPHIC GALOIS GROUP AND WEIL'S EXPLICIT FORMULAS

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A: **Galois group**: \( \Gamma_F = \text{Gal}(\overline{F}/F) = \varprojlim_{E} \text{Gal}(E/F) \), compact, total disconnected group.

B: **Weil group**: \( W_F = W_{E/F} = \varprojlim_{E} W_{E/F} \), locally compact group, with \( W_F \rightarrow \Gamma_F \).

C: **Langlands group**: hypothetical loc. compact group (Langlands/Kimura), with \( L_F \rightarrow W_F \), that would classify automorphic representations.
These groups have analogues for local completions $F_v$, with cong. classes of embeddings

$$
\begin{array}{ccc}
L_{F_v} & \longrightarrow & W_{F_v} \\
\downarrow & & \downarrow \\
L_F & \longrightarrow & W_F \\
\end{array} \quad \Gamma_{F_v} \quad \Gamma_F
$$

\text{(hypothetical)}

$n \in \text{val}(F)$,

\begin{equation}
L_{F_v} = \begin{cases} 
W_{F_v}, & \text{if } v \text{ is archimedean,} \\
W_{F_v} \times SU(2), & \text{if } v \text{ nonarchimedean}
\end{cases}
\end{equation}
Suppose $G/F$ reductive, $\pi \in \mathcal{T}_{\text{aut}}(G)$ (aut. rep of $G(F)$). We get a family

$$c(\pi) = \left\{ c_\nu(\pi) = c(\pi_\nu) : \nu \notin S \right\}, \quad \pi = \bigotimes \nu \pi_\nu,$$

of s.s. conj. classes in $L_G = \hat{G} \times \mathcal{W}_F$. Let

$$\mathcal{C}_{\text{aut}}(G) = \left\{ c(\pi) : \pi \in \mathcal{T}_{\text{aut}}(G) \right\}$$

be the set of equiv. classes of such families.

(while $c' \sim c$ if $c'_\nu = c_\nu$ for almost all $\nu$)
**Principle of Functoriality** (Conjecture of Langlands)

Suppose that $G, G' / F$ are quasisplit, and

$$\rho : L^{G'} \rightarrow L^G$$

is an $L$-homomorphism (i.e., an analytic homomorphism over $WF$). Then if

$$c' = \{ c_i' \} \text{ lies in } G_{\text{aut}}(G'),$$

its image

$$c = c(\rho') = \{ \rho(c_i') \}$$

lies in $G_{\text{aut}}(G)$.

If $r : L^G \rightarrow G_L(N, C)$, and $c \in G_{\text{aut}}(G)$, define partial $L$-function

$$L^S(s, c, r) = \prod_{v \notin S} \det(1 - r(cc_v)|_W, 1 - c_v)^{-1}.$$ 

**Functoriality** $\Rightarrow$ $L^S(s, c, r)$ has an analytic continuation and functional equation.

Assume functoriality from now on.
Candidate for $\mathcal{L}_F$: 2 ingredients

(I) Indexing set $\mathcal{S}_F$.

(II) For any $c \in \mathcal{S}_F$, an ext 

$$1 \rightarrow K_c \rightarrow L_c \rightarrow WF \rightarrow 1$$

of $WF$ by a compact, conn., simp. conn. gp $K_c$.

(I) If $G/F$ is a q. split, simple, s.c. gp, define $\mathfrak{g}_{\text{prim}}(G)$ to be the set of $c \in \text{Caut}(G) \cap \mathfrak{g}$ finite dim. rep. $r: L^2_G \rightarrow \text{End}(c)$

- and $\mathfrak{g}_{\text{prim}}(G)$ is the set of iso $\text{iso}^{\text{prim}}$ classes of pairs $(G, \mathcal{L})$

$$\mathcal{S}_F = \{(G, c) : c \in \mathfrak{g}_{\text{prim}}(G)\}$$
(II) Suppose \((G, c) \in C_F\). Let

\[(*) \quad 1 \rightarrow Z \xrightarrow{\varepsilon} \hat{G} \rightarrow G_{ad} \rightarrow 1\]

be a \(Z\)-extension of \(G_{ad}\). Then \(G = \hat{G}_{ad} \subset \hat{G}\).

E.g.: \(G = SL(m), \ G_{ad} = PGL(m), \ \hat{G} = GL(m), \ Z = G_m\).

Elem. hypothesis: \(\exists \ \varepsilon \in Aut(\hat{G}) \) whose fundamental image
under the dual map \(\hat{G} \rightarrow \hat{G}_c \) is the given \(c \in \text{Spim}(G)\).

Let

\[(\ast) \quad 1 \rightarrow \hat{G}_{sc} \rightarrow \hat{G} \xrightarrow{\varepsilon} \hat{Z} \rightarrow 1\]

be the dual exact seq. of \((*)\), and define
(i) \( \hat{c}(\hat{\xi}) \in \text{End}(\hat{\mathbb{G}}) (= c(\hat{\xi}) \), \( \hat{\xi} \) the central char of any \( \pi \in \Omega_{\text{Ad}}(\hat{\mathbb{G}}) \) with \( c(\hat{\xi}) = c \).

(ii) A dual 1-cocycle \( z_c : W_F \to \hat{\mathbb{G}} \) (Langlands' global class\(^\circ\) for toms \( \mathbb{Z} \).

(iii) A compact real form \( \hat{K}_c \) of \( \hat{G}_{sc} \), with normalization \( \hat{K}_c \) in \( \hat{\mathbb{G}} \).

(iv) \( L_c = \{ g \times w \in \hat{K}_c \times W_F : \hat{c}(g) = z_c(w) \} \).

This gives the long exact sequence:

\[ 1 \to K_c \to L_c \to W_F \to 1. \]

Given the ingredients (I)-(II), we define

\[ L_F = \prod_{\hat{c} \in \hat{G}_F} (L_c \to W_F) \]

as fibre product over \( W_F \).
Local Hypothesis: Local Langlands correspondence holds: more precisely,

\[ \forall (G, \varepsilon) \in \mathcal{S}_F \mapsto (\hat{G}, \varepsilon) \mapsto \hat{\pi} \in \hat{\Pi} \text{aut}(\hat{G}) \text{ with } c(\hat{\pi}) = \varepsilon, \]

then \( \forall v \in \text{val}(F), \) \( \hat{\pi}_v \in \hat{\Pi}(\hat{G}_v) \mapsto \hat{\phi}_v : L_{F_v} \rightarrow \hat{G}_v \) and \( \phi^1_v \in H^1(L_{F_v}, \hat{\mathfrak{g}}_v) \)

such that \( \hat{\phi}_v \) and \( \hat{\phi}^1_v \) depend only on \( \varepsilon \) (+ not \( \hat{\pi}_v \)).

Functoriality \( \Rightarrow \) Generalized Ramanujan \( \Rightarrow \) \( \text{im}(\hat{\phi}^1_v) \subset \hat{R}_v \)

(Langlands: Bochner article SLN 170, p. 43 - 48)

Since \( \hat{\phi}^1_v \) projects to localization \( z_{c,v} \in H^1(W_{F_v}, \hat{\mathbb{Z}}) \) of \( z_c, \)

\( \hat{\phi}^1_v (L_{F_v}) \subset \{ g_v \times w_v \in \hat{R}_c \times W_{F_v} : \hat{\varepsilon}(g_v) = z_{c,v}(w_v) \} \subset L_c, \)

so that \( \hat{\phi}^1_v : L_{F_v} \rightarrow L_c. \) The fibre product \( LF \)

thus comes with embeddings \( L_{F_v} \rightarrow LF. \)
Conclusion: We have constructed an explicit loc. cp^§ group

\[ LF \longrightarrow WF, \]

with the conj. classes of local embeddings

\[ L_{F_n} \longrightarrow W_{F_n} \longrightarrow \Gamma_{F_n} \]

\[ \downarrow \quad \downarrow \quad \downarrow \]

\[ L_F \longrightarrow WF \longrightarrow \Gamma_F, \quad \text{eval}(F). \]
Problem: Formulate a slightly broader hypothesis (Functoriality Plus) which

(i) would include functoriality + 2 hypotheses above

(ii) would be a necessary part of proof of functoriality

proposed by Langlands in *Beyond Endoscopy*, and

(iii) would imply that $L\mathcal{F}$ is the Langlands group —

i.e. that it comes with a canonical bijection

$$\{ \phi : L\mathcal{F} \to GL(N, \mathbb{C}), \text{irred, unitary} \} \leftrightarrow \{ \pi \in \Pi_{\text{cusp, unit}}(GL_N) \}$$

compatible with localizations $\phi_v$ and $\pi_v$. 
Weil's explicit formula (see also Shim - Templier)

Define \( C_c^0(L_F) \) - space of \( f_{m^S} \) that descend to \( C_c^0(\mathfrak{m} \backslash L_F) \), for normal sub\( \mathfrak{m} \) \( \mathfrak{m} = \mathfrak{m}_S \) of \( \text{finite codim in } L_F \) - i.e. \( \mathfrak{m} \backslash L_F \) is algebraic over \( \mathbb{Q} \).

Define \( L_F = \ker (x \to |x|_1) \) - compact \( \sigma_F \), where \( |x|_1 \) is the abs. value of image of \( x \in L_F \) in \( W_F \).

\[ r : L_F \to GL(N, \mathbb{C}) \]

is an \( N \)-dim. rep. of \( L_F \), so is

\[ r_2(x) = r(x) |x|_1^2, \quad \mathbb{C} \in C. \]

Given \( r \), we can form

\[ L(2, r) = \bigotimes_{v} L(2, r_v) = L_S(2, r) L_S(2, r), \]

where \( r \) is "unramified" outside \( S = S_F \cup S_{\infty} \). Since we are assuming functoriality, \( L(2, r) \) has an cont. \& \text{final} eq \( \approx \).
Define \( \mathcal{Z}_F(r) = \{ \rho \in \mathcal{C} : L(\rho, r) = 0 \} \).

For any \( f \in C_c^\infty(\mathcal{L}_F) \), the sum
\[
\sum_{\rho \in \mathcal{Z}_F(r)} \text{tr}(\rho(f))
\]
depends only on \( \text{net}^{\infty} \) of \( r \) to \( \mathcal{L}_F \). We can then form the "spectral sum"
\[
\sum_{r \in \text{net}^{\infty} \mathcal{L}_F} \sum_{\rho \in \mathcal{Z}_F(r)} \text{tr}(\rho(f)), \quad f \in C_c^\infty(\mathcal{L}_F),
\]
where \( \text{net}^{\infty} \mathcal{L}_F \) is set of equiv. classes of irred. (finite dim.) reps of \( \mathcal{L}_F \).
Given $f \in C_c^\infty(x_f \backslash L_F) \subset C_c^\infty(L_F)$, define set

$$S = S_f \subset S_\infty$$

of $\text{vol}(F) \cdot \mathfrak{z}$. If $\mathfrak{z} \neq S$, $S_f$ contains the inertia subgroup $I_{F,\mathfrak{z}} \times SU(2) \subset L_{F,\mathfrak{z}} \subset L_F$.

and hence gives a Frobenius conjugacy class $c = c_\mathfrak{z}$ in $\mathfrak{z} \backslash L_F$.

We can then form the "orbital integral"

$$\int_{L_{F,c} \backslash L_F} f(x^{-1} \mathfrak{z} x) \, dx,$$

and the "geometric sum"

$$\sum_{c \in C_F} \sum_{m \in \mathbb{N}} \text{vol}(C_{F,c}^S \backslash L_{F,c}) \int_{L_{F,c} \backslash L_F} f(x^{-1} c^m x) \, dx,$$

where

$$C_F^S = \{ c = c_\mathfrak{z} : \mathfrak{z} \cap S = S_f \},$$

and $C_{F,c}^S = \{ c^k : k \in \mathbb{Z} \} \subset L_{F,c}$. 
**THEOREM**: Suppose that $f$ is symmetric, in the sense that $f(w) = f(x) = f(x^{1}) |x|$. Then the "geometric side"

$$
- \sum_{c \in C_{F}} \sum_{m \in \Gamma_{N}} \text{vol}(C_{F, c \setminus L_{F, c}}) \int_{L_{F, c \setminus L_{F}}} f(x^{1} e^{m} x) \, dx
$$

$$
+ \sum_{\text{ret}(L_{F})} \frac{1}{2 \pi i} \sum_{\text{re}(z) = \frac{1}{2}} \frac{L_{F}(z, r)}{L_{F}(z, r)} \text{tr}(\rho_{\zeta}(\bar{c}))
$$

equals the "spectral side"

$$
\frac{1}{2} \sum_{\text{ret}(L_{F})} \sum_{\rho \in \text{rep}(\overline{\Gamma}(r))} \text{tr}(\rho_{\zeta}(\bar{c})) - \int_{L_{F}} f(x) \, dx.
$$