Introduction to Proofs
IAP 2015
Lecture Notes 3

1. Comparing sets I: countability and uncountability

We now turn our attention to the goal of understanding some comparisons between the "relative size" of various sets. To make our first instance of this notion precise, we shall use the following definition:

Definition. *A nonempty set A is said to be countable if there exists an injectve map $\sigma : A \to \mathbb{N}$.*

Our first remark on this notion of countability is that a set A is countable if and only if there exists a surjection $\tau : \mathbb{N} \to A$. To see that this holds, we will make use of a preliminary claim (to be shown in homework):

Proposition 1.1. *Let A and B be sets, and let $f : A \to B$ be a given function. Then the function f is injective if and only if there exists a function $g : B \to A$ which is a left inverse for f in the sense that one has the identity $g(f(x)) = x$ for all $x \in A$.*

Moreover, f is surjective if and only if there exists a function $g : B \to A$ which is a right inverse for f in the sense that one has $f(g(x)) = x$ for all $x \in B$.

We now return to the statement that a set A is countable if and only if there exists a surjection $\tau : \mathbb{N} \to A$ (indeed, these two criteria are *equivalent*, and we could have equally well taken the "surjection criterion" as our primary definition for countability). We give the argument for each direction of the "if and only if" individually:

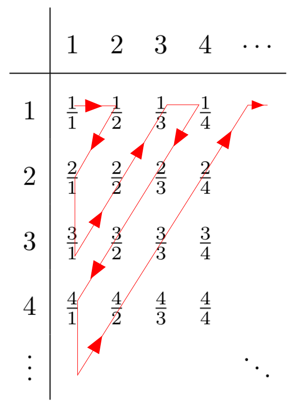
($\Rightarrow$): This portion of the claim states that if a nonempty set A is countable, then there exists a surjection $\tau : \mathbb{N} \to A$. Suppose that A is given as a (nonempty) countable set; we may therefore find an injection $\sigma : A \to \mathbb{N}$. Choose some element $a \in A$, and define a map $\tau : \mathbb{N} \to A$ by $\tau(n) = a$ if $\sigma^{-1}(\{n\}) = \emptyset$, and $\tau(n)$ as the singleton element of $\sigma^{-1}(\{n\})$ if this set is nonempty. The proof concludes by observing that τ is surjective: if $x \in A$ is a given element of the set A, we can set $n = \sigma(x) \in \mathbb{N}$ to obtain $\tau(n) = x$ by construction.

($\Leftarrow$): Suppose that A is a nonempty set with a surjection $\tau : \mathbb{N} \to A$. It follows from the proposition that τ has a right inverse $\sigma : A \to \mathbb{N}$ (in the present setting we can in fact give a precise construction – choose $\sigma(a) = \min \tau^{-1}(\{a\})$ for each $a \in A$). It remains to show that σ is injective, for which we argue as follows: let $a, b \in A$ be given such that $\sigma(a) = \sigma(b)$. We then have $a = \tau(\sigma(a)) = \tau(\sigma(b)) = b$, so that $a = b$. Since a and b were arbitrary, σ is injective as desired. (This is an instance of one direction of the above proposition: τ is a left inverse for σ, so that σ must be injective.)

We now recall two basic examples which show how to work with the notion of countability.

Example 1.2 ($\mathbb{Q}$ is countable). We give the idea of an argument to show that $\mathbb{Q}$ is countable. By the remarks above, it suffices to find a surjection $\tau : \mathbb{N} \to \mathbb{Q}$. For this purpose, we will arrange the rational numbers in a table as follows: for each $m, n \geq 1$, let the rational number $\frac{m}{n}$ be placed in the mth row and nth column of the table. We may then construct the function τ by assigning $\tau(1) = \frac{1}{1}$ and counting through diagonal paths of the table (for instance by the path indicated in the diagram below; other choices of enumeration are certainly possible as well), assigning $\tau(\ell)$ to the ℓth entry encountered along the path. Since this collection of paths eventually encounters every entry in the table, and every element of $\mathbb{Q}$ appears in the table at least once, the resulting map τ is surjective.

	1	2	3	4	$\cdots$
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	
2	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	
3	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	
4	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	
$\vdots$					$\ddots$



Remark 1.3. The above discussion is not quite a rigorous proof, since the precise "path" through the table is not clearly specified; however, this can be filled in with a small amount of extra work.

Example 1.4 ($\mathbb{R}$ is not countable). We sketch an argument which shows the slightly modified claim that the interval $(0, 1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is not countable. Suppose for contradiction that it were; then we can find a surjective map $\tau : \mathbb{N} \to \mathbb{R}$. For each $(n, k) \in \mathbb{N} \times \mathbb{N}$ let $x_{n,k}$ denote the kth digit after the decimal point in the decimal expansion of $\tau(n)$. Define

$$\widetilde{x}_{n,k} = \begin{cases} (x_{n,k}) + 1, & \text{if } x_{n,k} < 9, \\ 0, & \text{otherwise,} \end{cases}$$

and let x_* be the limit of the sequence

$$\sum_{n=1}^{N} \frac{\widetilde{x}_{n,n}}{10^n}$$

as $N \to \infty$.

2. Working with real numbers

2.0.1. *Properties of real numbers.* In our discussion below, we will use the following important property of real numbers: for every nonempty $A \subset \mathbb{R}$ which is bounded from above in the sense that there exists $M \in \mathbb{R}$ such that $x \leq M$ for all $x \in A$ there exists a number $S \in \mathbb{R}$ (often denoted as the *supremum* $\sup A$) with the property that

(i) $x \leq S$ for every $x \in A$ (S is an "upper bound" for A), and

(ii) For every $T \in \mathbb{R}$ satisfying the condition "$x \leq T$ for all $x \in A$", one has $S \leq T$. (S is the *smallest* upper bound for A).

The value S is uniquely determined by these properties.

Similarly, for every nonempty $A \subset \mathbb{R}$ which is bounded from below in the sense that there exists $N \in \mathbb{R}$ such that $x \geq N$ for all $x \in A$, there exists $L \in \mathbb{R}$ (often denoted as the *infimum* $\inf A$) such that

(i') $x \geq L$ for every $x \in A$ (L is a "lower bound" for A), and
(ii') For every $T \in \mathbb{R}$ satisfying the condition "$x \geq L$ for all $x \in A$", one has $L \leq T$. (L is the *largest* lower bound for A).

The value L is also uniquely determined.

Remark 2.1. Note that one can reduce the construction of the infimum of a set A to the corresponding result for the supremum by considering $A' = \{-x : x \in A\}$ and setting

$$L = -\sup A'.$$

Exercise 2.2. Let $A \subset \mathbb{R}$ be a set which is bounded from above. Show that S satisfying the above conditions is uniquely determined. That is, show that if $S, S' \in \mathbb{R}$ both satisfy (i) and (ii), then $S = S'$.
Hint: Let S, S' be as stated, and show that both $S \leq S'$ and $S' \leq S$ hold.

Example 2.3. Consider the set $A = \{x \in \mathbb{R} : x^2 < 2\}$. The set A is bounded from above, as a consequence of the observation that the function from $(0, \infty)$ to $\mathbb{R}$ given by the rule $x \mapsto x^2$ is increasing (this follows from the ordering properties of $\mathbb{R}$ – essentially, these are the usual properties of the inequality symbols, and we will use them without further comment).

2.1. **Limits.** Let $a, b \in \mathbb{R}$ be given with $a < b$, and let f be a function mapping (a, b) into $\mathbb{R}$. For each $c \in (a, b)$ and $F \in \mathbb{R}$, we say that the limit

$$\lim_{x \to c} f(x) = F$$

holds if the following condition is satisfied:

For every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in (a, b)$,
$0 < |x - c| < \delta$ implies $|f(x) - F| < \epsilon$.

Similarly, we say that the limit

$$\lim_{x \to c} f(x)$$

exists (alternatively, $x \mapsto f(x)$ has a limit as x approaches c) if there exists $F \in \mathbb{R}$ such that the above condition is satisfied.

To see how to work with this definition in practice, we now examine an example of a common style of argument used to show that a particular limit exists.

Example 2.4. Show that $\lim_{x \to x_0} x^2 = x_0^2$ for all $x_0 \in (0, 1)$.

Proof. Let $x_0 \in (0, 1)$ and $\epsilon > 0$ be given. Set $\delta = \min\{1, \frac{\epsilon}{3}\}$. Let $x \in \mathbb{R}$ be given such that $|x - x_0| < \delta$. We then have

$$|x^2 - (x_0)^2| = |x + x_0| \cdot |x - x_0|$$

$$
\begin{aligned}
&< (|x - x_0| + |2x_0|) \cdot \delta \\
&< \delta^2 + 2\delta \\
&\leq 3\delta
\end{aligned}
$$

where we have used $0 < \delta \leq 1$ to ensure $\delta^2 \leq \delta$. Now, using $\delta \leq \frac{\epsilon}{3}$ we obtain

$$|x^2 - (x_0)^2| < \epsilon.$$

Since the value of x was arbitrary, this concludes the proof of the desired limit. $\square$

2.2. **Continuity.** We now address the definition of continuity of functions. Let $a, b \in \mathbb{R}$ again be given with $a < b$ and let $f : (a, b) \to \mathbb{R}$ be a function. We say that f is continuous at a point $c \in (a, b)$ if the following condition holds:

For every $\epsilon > 0$ there exists $\delta = \delta(\epsilon, c) > 0$ such that for all $x \in (a, b)$, $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$.

We say that f is continuous on (a, b) if f is continuous at c for every $c \in (a, b)$.

In analogy with Example 2.4 above, the next exercise gives some basic practice with manipulating the definiton of continuity.

Exercise 2.5. Let $f : (0, 1) \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be the functions given by

$$x \overset{f}{\mapsto} \sqrt{x} \quad \text{and} \quad x \overset{g}{\mapsto} x^2.$$

Show that f and g are continuous on their domains.

The solution makes use of the strategy of Example 2.4. Letting c be an arbitrary element of the domain and letting $\epsilon > 0$ be given, our goal is to find $\delta > 0$ satisfying the condition

$$|x - c| < \delta \quad \Rightarrow \quad |f(x) - f(c)| < \epsilon \tag{1}$$

from the definition of continuity. Once this choice is made, we have to demonstrate that the condition (1) is indeed satisfied. This is accomplished by letting x be an arbitrary element of the domain satisfying $|x - c| < \delta$, and showing that this implies $|f(x) - f(c)| < \epsilon$. We give the precise argument for the function f below:

Proof of continuity of f in Exercise 2.5. We first show the continuity of f. Let $c \in (0, 1)$ and $\epsilon > 0$ be given. Set $\delta = \epsilon\sqrt{c}$. Now, let $x \in (0, 1)$ be given with $|x - c| < \delta$. We then have

$$
\begin{aligned}
|f(x) - f(x)| = |\sqrt{x} - \sqrt{c}| &= \left| \frac{x - c}{\sqrt{x} + \sqrt{c}} \right| \\
&< \frac{\delta}{\sqrt{x} + \sqrt{c}} < \frac{\delta}{\sqrt{c}} = \epsilon.
\end{aligned}
$$

Since $\epsilon > 0$ was arbitrary, we conclude that f is continuous at c as desired. $\square$