Introduction to Proofs
IAP 2015
In-class problems for day 8

We say that a set $A \subset \mathbb{R}$ has measure zero if for every $\epsilon > 0$ there exists a sequence of pairs of real numbers

$$\Big((a_n, b_n)\Big)_{n \geq 1} \text{ with } (a_n, b_n) \in \mathbb{R}^2 \text{ and } a_n < b_n \text{ for all } n \geq 1,$$

such that

$$A \subset \bigcup_{n \geq 1} [a_n, b_n]$$

and

$$\sum_{n \geq 1} b_n - a_n < \epsilon.$$

Problem 13. *Suppose that $A \subset \mathbb{R}$ is a countable set. Then A has measure zero.*

Proof.

□

Problem 14. *Suppose that $(A_i : i \geq 1)$ is a sequence of subsets of $\mathbb{R}$ such that each A_i has measure zero. Then*

$$\bigcup_{i \geq 1} A_i$$

also has measure zero.

Proof.

□

Problem 15. *In this problem, we will construct the* **Cantor set***, an uncountable set which has measure zero.*

The construction will be based on a recursively defined sequence of sets, each of which consists of a union of intervals. Begin by setting

$$C_0 = [0,1].$$

Suppose now that the set C_n for some $n \geq 0$ can be written as

$$C_n = I_1 \cup I_2 \cup \cdots \cup I_m$$

for some $m \in \mathbb{N}$, where each I_j, $j = 1, \cdots, m$, is a closed interval $[a,b]$ ($a, b \in \mathbb{R}$, $a < b$).

For each $I = [a,b] \subset \mathbb{R}$, define

$$R_3(I) = \left[a, a + \left(\frac{b-a}{3}\right)\right] \cup \left[b - \left(\frac{b-a}{3}\right), b\right].$$

We now set

$$C_{n+1} = \bigcup_{j=1}^{m} R_3(I_j).$$

This defines the sequence (C_n) *recursively.* The *Cantor set* is then

$$C := \bigcap_{n \geq 1} C_n.$$

(1) Draw a series of diagrams to visualize the construction of the sequence $(C_n : n \geq 1)$.

(2) Show that C has measure zero.

Proof.

$\square$

(3) It can be shown that C is equal to the set $\{\sum_{n\geq 1} a_n 3^{-n} : a_n \in \{0,2\}\}$ consisting of "base 3 expansions." Use this to show that C is uncountable (hint: mimic our proof of uncountability of the interval $(0,1) \subset \mathbb{R}$).

Proof.

$\square$