Introduction to Proofs
IAP 2015
In-class problems for day 4

Problem 8. *Let* $(A_i : i \geq 1)$ *be a sequence of countable sets. Show that*

$$\bigcup_{i \in \mathbb{N} \setminus \{0\}} A_i$$

is also countable.

Proof. For each $i \in \mathbb{N} \setminus \{0\}$, the countability of A_i implies that we may choose a surjective function

$$\tau_i : \mathbb{N} \to A_i.$$

Moreover, using the countability of $\mathbb{N} \times \mathbb{N}$ (which we proved in class), we can find a surjective map $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. For each $n \in \mathbb{N}$, we may then write

$$f_1(n) = \pi_1(f(n)), \quad f_2(n) = \pi_2(f(n))$$

where $\pi_1, \pi_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ are defined by

$$\pi_1((a,b)) = a, \quad \pi_2((a,b)) = b$$

for $(a,b) \in \mathbb{N} \times \mathbb{N}$.

To show that

$$X := \bigcup_{i \in \mathbb{N} \setminus \{0\}} A_i$$

is countable, we define a map

$$\tau_* : \mathbb{N} \to X$$

by setting

$$\tau_*(n) = \tau_{f_1(n)}(f_2(n)).$$

Since $\tau_i(\mathbb{N}) \subset A_i \subset X$ holds for each $i \in \mathbb{N} \setminus \{0\}$, the map is well defined.

To show that X is countable, it remains to show that τ_* is surjective. For this, let $x \in X$ be given. Then (by the definition of the union) there exists $i_* \in \mathbb{N} \setminus \{0\}$ such that x belongs to the set A_{i_*}. Since τ_{i_*} is surjective, we can then find $j_* \in \mathbb{N}$ with

$$\tau_{i_*}(j_*) = x.$$

We now apply the surjectivity of f to find $n_* \in \mathbb{N}$ such that

$$f(n_*) = (i_*, j_*).$$

The definition of τ_* now gives

$$\tau_*(n_*) = \tau_{f_1(n_*)}(f_2(n_*)) = \tau_{i_*}(j_*) = x.$$

Since $x \in X$ was arbitrary, we conclude that τ_* is surjective as desired. $\square$