Introduction to Proofs
IAP 2015
In-class problems for day 4

Problem 7. *Use mathematical induction to show that for every $n \in \mathbb{N}$, the quantity*

$$(3+\sqrt{5})^n + (3-\sqrt{5})^n$$

is an even integer.

Proof. Note that the assertion is trivial for $n = 0$. We use induction on $n \geq 1$ to show the stronger claim that for all $n \geq 1$, the quantity

$$(3+\sqrt{5})^n + (3-\sqrt{5})^n \tag{1}$$

is an even integer and

$$(3+\sqrt{5})^n - (3-\sqrt{5})^n = 2k\sqrt{5} \tag{2}$$

for some $k \in \mathbb{N}$.

We begin by showing the base case, $n = 1$. For this, we observe

$$(3+\sqrt{5}) + (3-\sqrt{5}) = 6$$

and

$$(3+\sqrt{5}) - (3-\sqrt{5}) = 2\sqrt{5},$$

from which the desired claim is immediate.

We now show the inductive step. Suppose that the claim holds with $n = \ell$ for some $\ell \geq 1$. We want to show the claim for $n = \ell + 1$. For this, we write

$$\begin{aligned}(3+\sqrt{5})^{\ell+1} + (3-\sqrt{5})^{\ell+1} &= (3+\sqrt{5})^{\ell}(3+\sqrt{5}) + (3-\sqrt{5})^{\ell}(3-\sqrt{5}) \\ &= 3\Big((3+\sqrt{5})^{\ell} + (3-\sqrt{5})^{\ell}\Big) \\ &\quad + \sqrt{5}\Big((3+\sqrt{5})^{\ell} - (3-\sqrt{5})^{\ell}\Big).\end{aligned}$$

Using the inductive hypothesis, we conclude that the first term in this expression is an even integer, while the second term is equal to

$$\sqrt{5}(2k\sqrt{5}) = 10k$$

for some $k \in \mathbb{N}$. Thus, (1) is an even integer as desired.

It remains to show that (2) can be written in the form $2k\sqrt{5}$ for some $k \in \mathbb{N}$. To obtain this, we note that

$$\begin{aligned}(3+\sqrt{5})^{\ell+1} - (3-\sqrt{5})^{\ell+1} &= (3+\sqrt{5})^{\ell}(3+\sqrt{5}) - (3-\sqrt{5})^{\ell}(3-\sqrt{5}) \\ &= 3\Big((3+\sqrt{5})^{\ell} - (3-\sqrt{5})^{\ell}\Big) \\ &\quad + \sqrt{5}\Big((3+\sqrt{5})^{\ell} + (3-\sqrt{5})^{\ell}\Big) \\ &= 3(2k\sqrt{5}) + 2m\sqrt{5}\end{aligned}$$

for some $k, m \in \mathbb{N}$, where we have made use of the inductive hypothesis to obtain the last line. This quantity is therefore equal to $2(3k+m)\sqrt{5}$, which is of the form $2k'\sqrt{5}$, $k' \in \mathbb{N}$, as desired. $\square$