Introduction to Proofs
IAP 2015
Solutions to in-class problems for day 3

Problem 4. *Use induction to show that the identity*

$$\frac{1-a^n}{1-a} = \sum_{k=0}^{n-1} a^k$$

holds for every $a \in \mathbb{R} \setminus \{1\}$ and all $n \in \mathbb{N}$.

Proof. Let $a \in \mathbb{R} \setminus \{1\}$ be given. We show the claim by induction on $n \in \mathbb{N}$.

(Base cases, $n = 0$ and $n = 1$): In the case $n = 0$, the claim becomes $0 = 0$ (in view of the identity $1 - a^0 = 1 - 1 = 0$, and since there are no terms contributing to the sum on the right-hand side). In the case $n = 1$, we have $\frac{1-a^1}{1-a} = 1$, so that the claim becomes $1 = a^0$, which is again immediately true.

(Inductive step): Suppose that the claim holds for some $n \geq 1$.[1] We want to show that it is also valid with n replaced by $n + 1$, i.e. that the identity

$$\frac{1-a^{n+1}}{1-a} = \sum_{k=0}^{n} a^k \tag{1}$$

holds. Note that, in view of the induction hypothesis, we have

$$\sum_{k=0}^{n} a^k = \left(\sum_{k=0}^{n-1} a^k\right) + a^n = \frac{1-a^n}{1-a} + a^n.$$

To conclude our argument, we now observe that the right hand side is equal to

$$\frac{1-a^n + a^n(1-a)}{1-a} = \frac{1-a^{n+1}}{1-a}.$$

This gives the equality (4) as desired. □

Problem 5. *Show that for every $n \in \mathbb{N}$, if $2^n - 1$ is prime, then n is prime.*

Hint: Use the result of Problem 4 above.

Proof. We show the contrapositive, that if $m \in \mathbb{N}$ is not prime, then $2^m - 1$ is not prime. We therefore let $m \in \mathbb{N}$ be given such that m is not a prime number. We can then find two integers $r, s \in \mathbb{N}$ with $1 < r < m$ and $1 < s < m$ such that $m = rs$.

On the other hand, invoking the result of Problem 4 (with $a = 2^r$ and $n = s$), we have

$$\frac{2^m - 1}{2^r - 1} = \frac{(2^r)^s - 1}{2^r - 1} = \sum_{k=0}^{s-1} 2^{rk},$$

and thus

$$2^m - 1 = (2^r - 1)\left(\sum_{k=0}^{s-1} 2^{rk}\right).$$

[1] This is our *induction hypothesis.*

Since the expressions $2^r - 1$ and $\sum_{k=0}^{s-1} 2^{rk}$ are each integers, to conclude that $2^m - 1$ is not prime, it suffices to show $2^r - 1 > 1$ and $2^r - 1 < 2^m - 1$. The first of these inequalities follows by observing that $r > 1$ implies $2^r > 2$, and thus $2^r - 1 > 1$ holds, while the second follows by noting that $r < m$ implies $2^r - 1 < 2^m - 1$ as desired. □

Problem 6. *We showed in class that* $e := \sum_{k=0}^{\infty} \frac{1}{k!}$ *is irrational. Prove the stronger result that* e^2 *is irrational by using the series*

$$\frac{1}{e} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}.$$

Question: Why do we say that this is a "stronger" result?

Hint: If $e^2 = \frac{a}{b}$*, write* $be = \frac{a}{e}$*.*

Proof. Following the hint, suppose for contradiction that we have

$$e^2 = \frac{a}{b}$$

for some (positive) integers a and b, and write

$$be = \frac{a}{e}.$$

Repeating the argument used in the proof of [Example 2.5, Lecture Notes 2], for all $k \in \mathbb{N}$ with $k > 2b$ we have

$$k!be = b\left(\sum_{\ell=0}^{k} \frac{k!}{\ell!}\right) + k!b\left(\sum_{\ell=k+1}^{\infty} \frac{1}{\ell!}\right) =: (B)_1 + (B)_2,$$

with

$$(B)_1 \in \mathbb{Z}$$

and

$$0 < (B)_2 < \frac{1}{2}. \tag{2}$$

Similarly, we have, for all $k \in \mathbb{N}$,

$$\frac{k!a}{e} = a\left(\sum_{\ell=0}^{k} \frac{(-1)^\ell k!}{\ell!}\right) + k!a\left(\sum_{\ell=k+1}^{\infty} \frac{(-1)^\ell}{\ell!}\right) =: (A)_1 + (A)_2.$$

with

$$(A)_1 \in \mathbb{Z}$$

and (using the triangle inequality)

$$|(A)_2| \leq k!a \sum_{\ell=k+1}^{\infty} \frac{1}{\ell!} \leq \sum_{\ell=k+1}^{\infty} \frac{a}{(k+1)^{\ell-k}} = \frac{a}{k},$$

so that for $k > 2a$ we have

$$0 \leq |(A)_2| < \frac{1}{2}. \tag{3}$$

Moreover, if k is even, we obtain

$$(A)_2 = -\frac{a}{k+1} + \sum_{\ell=k+2}^{\infty} \frac{(-1)^\ell k!a}{\ell!}$$

$$
\begin{aligned}
&\leq -\frac{a}{k+1} + \sum_{\ell=k+2}^{\infty} \frac{k!a}{\ell!} \\
&\leq -\frac{a}{k+1} + \sum_{\ell=k+2}^{\infty} \frac{a}{(k+1)^{\ell-k}}. \qquad (4)
\end{aligned}
$$

Evaluating the geometric series appearing on the right-hand side of (4) gives

$$
\begin{aligned}
(A)_2 &\leq -\frac{a}{k+1} + \frac{a}{(k+1)^2}\left(\frac{1}{1-\frac{1}{(k+1)}}\right) \\
&= -\frac{a(k-1)}{k(k+1)} \\
&< 0.
\end{aligned}
$$

Recalling that

$$(B)_1 + (B)_2 = (A)_1 + (A)_2$$

holds for all $k \in \mathbb{N}$ by construction, we conclude that for all even $k \in \mathbb{N}$ with $k > \max\{2a, 2b\}$, we have

$$0 < (B)_2 - (A)_2 < 1$$

(since, $(B)_2 - (A)_2 \leq |(B)_2| + |(A)_2| \leq \frac{1}{2} + \frac{1}{2}$ follows from (2) and (3)). On the other hand, we also have

$$(B)_2 - (A)_2 = (A)_1 - (B)_1 \in \mathbb{Z}$$

for all such k. Since we cannot simultaneously have $(B)_2 - (A)_2 \in (0,1)$ and $(B)_2 - (A)_2 \in \mathbb{Z}$, this gives the desired contradiction. Thus, no choice of a and b as stated above is possible, and we conclude that e^2 is irrational as desired. $\square$

Remark. As in our discussion of Example 2.5 in Lecture Notes 2, we have (again, somewhat loosely) used several properties of convergent (and absolutely convergent) series in the proof. We postpone further discussion of this point.