Introduction to Proofs
IAP 2015
Solution to first in-class problem for day 2

Problem 2. *Let X and Y be sets, and let $f : X \to Y$ be a given function. Show that for all $A, B \subset Y$,*

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

and for all $U, V \subset X$,

$$f(U \cup V) = f(U) \cup f(V).$$

Hint: The result can be divided into four separate implications.

Proof. Let $A, B \subset Y$ be given. We first show that

$$f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B).$$

For this, let $x \in f^{-1}(A \cap B)$ be given, so that $f(x) \in A \cap B$, and therefore both of the conditions $f(x) \in A$ and $f(x) \in B$ hold. Rewriting these conditions, we obtain $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$, so that $x \in f^{-1}(A) \cap f^{-1}(B)$ as desired. Since $x \in f^{-1}(A \cap B)$ was arbitrary, this shows the desired implication.

We now prove

$$f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B).$$

Let $x \in f^{-1}(A) \cap f^{-1}(B)$ be given. This implies that both $f(x) \in A$ and $f(x) \in B$ hold, so that we have $f(x) \in A \cap B$. Thus $x \in f^{-1}(A \cap B)$ as desired.

Let $U \subset X$ and $V \subset X$ be given. We first demonstrate the inclusion

$$f(U \cup V) \subset f(U) \cup f(V).$$

For this, let $x \in f(U \cup V)$ be given. We can then choose $y \in U \cup V$ such that $x = f(y)$. If y belongs to the set U, then we immediately conclude $x \in f(U)$ (and, since $f(U) \subset f(U) \cup f(V)$ holds trivially, $x \in f(U) \cup f(V)$). On the other hand, if y does not belong to U, $y \in U \cup V$ implies that y belongs to V, and we therefore have $x \in f(V) \subset f(U) \cup f(V)$. Thus, in either case, we have $x \in f(U) \cup f(V)$, which implies the desired inclusion.

It remains to establish that

$$f(U) \cup f(V) \subset f(U \cup V).$$

Let $x \in f(U) \cup f(V)$ be given. Suppose first that $x \in f(U)$. It then follows that there exists $y \in U$ such that $x = f(y)$. Since $U \subset U \cup V$, we therefore have $y \in U \cup V$, and thus $x \in f(U \cup V)$. Alternatively, if $x \notin f(U)$, the condition $x \in f(U) \cup f(V)$ implies $x \in f(V)$, which gives the existence of some $y \in V \subset U \cup V$ with $x = f(y)$. This in turn implies $x \in f(U \cup V)$ as desired. Since $x \in f(U \cup V)$ holds in either case, the desired inclusion follows.

$\square$