18.S097 Introduction to Proofs
IAP 2015
Solution to Homework 5

Instructions: Choose and complete **one** of the following problems (you only have to do one!):

Problem 1. *Show that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for every open set $U \subset \mathbb{R}$, the set*

$$f^{-1}(U) = \{x \in \mathbb{R} : f(x) \in U\}$$

is also open.

Solution:

We begin by showing that $f : \mathbb{R} \to \mathbb{R}$ continuous implies that $f^{-1}(U)$ is open for all $U \subset \mathbb{R}$ open. Let f be a given continuous function on $\mathbb{R}$, and let $U \subset \mathbb{R}$ be a given open set. We want to show that $f^{-1}(U)$ is open; for this, let $x \in f^{-1}(U)$ be given. We then have $f(x) \in U$, and thus, since is U open, we can find $\delta_1 > 0$ such that $B(f(x); \delta_1) \subset U$. By the continuity of f, we can then find $\delta_2 > 0$ such that for every $x \in \mathbb{R}$, the condition

$$|y - x| < \delta_2 \tag{1}$$

implies

$$|f(y) - f(x)| < \delta_1. \tag{2}$$

Set $\delta = \delta_2$. We will show the inclusion $B(x; \delta) \subset f^{-1}(U)$. Let $y \in B(x; \delta)$ be given. By the definition of the set $B(x; \delta)$, this means that (1) holds, so that (2) holds as well (by the choice of δ_2). However, this condition can be rewritten as $f(y) \in B(f(x); \delta_1) \subset U$. Thus $y \in f^{-1}(U)$, and the desired inclusion holds. Since $x \in f^{-1}(U)$ was arbitrary, we have shown that $f^{-1}(U)$ is open, as desired.

We now show the converse implication, that the condition

$$f^{-1}(U) \text{ is open for all open } U \subset \mathbb{R} \tag{3}$$

implies f is continuous. Let $f : \mathbb{R} \to \mathbb{R}$ be a given function, and suppose that (3) holds. Let $x \in \mathbb{R}$ and $\epsilon > 0$ be given. Since $B(f(x); \epsilon)$ is an open set (recall that we showed in class that every open interval (a, b) is an open set; recall also that $B(f(x); \epsilon)$ is equal to the interval $(f(x) - \epsilon, f(x) + \epsilon)$),

$$f^{-1}(B(f(x); \epsilon))$$

is also open. Since $x \in f^{-1}(B(f(x); \epsilon))$, we can find $\delta > 0$ such that

$$B(x; \delta) \subset f^{-1}(B(f(x); \epsilon)).$$

It remains to show that this δ satisfies the requirements for the epsilon-delta definition of continuity. Let $y \in \mathbb{R}$ be given with $|y - x| < \delta$. We then have $y \in B(x; \delta)$, so that y also belongs to the set $f^{-1}(B(f(x); \epsilon))$, and thus

$$f(y) \in B(f(x); \epsilon).$$

This latter condition can be written as $|f(y) - f(x)| < \epsilon$. Since $x \in \mathbb{R}$ and $\epsilon > 0$ were arbitrary, f is continuous as desired.

Problem 2. *Let $(q_n)_{n\geq 1}$ be an enumeration of $\mathbb{Q}\cap(0,1)$ (so that $q_n \in \mathbb{Q}\cap(0,1)$ for each $n\geq 1$, and $\{q_n : n\geq 1\} = \mathbb{Q}\cap(0,1)$). Define*

$$A := \bigcup_{n\geq 1}\left(q_n - \frac{1}{2^{n+2}}, q_n + \frac{1}{2^{n+2}}\right).$$

Show that $A^c := (0,1)\setminus A$ is not a set of measure zero.

Solution: (sketch)

Let A be as stated. Suppose that A^c has measure zero; then, for each $\epsilon > 0$, we can find a collection of closed intervals $([a_m, b_m])_{m\geq 1}$ such that $A^c \subset \cup_{m\geq 1}[a_m, b_m]$ and $\sum_{m\geq 1} b_m - a_m < \epsilon$. We apply this condition with $\epsilon = \frac{1}{4}$ to choose such a sequence $([a_m, b_m])$.

It now follows from $(0,1) = A\cup A^c$ that $(0,1)$ is contained in the set

$$X = \bigcup_{\ell=1}^{\infty} I_\ell := \bigcup_{n\geq 1}[q_n - \frac{1}{2^{n+2}}, q_n + \frac{1}{2^{n+2}}] \cup \bigcup_{m\geq 1}[a_m, b_m],$$

where each I_ℓ is an interval appearing in the union on the right side of the last line.

We now obtain the desired contradiction. Define

$$\sigma := \inf\left\{\sum_{k=1}^{\infty} d_k - c_k : ([c_k, d_k])_{k\geq 1} \text{ is s.t. } X \subset \bigcup_{k\geq 1}[c_k, d_k]\right\}.$$

(We note that the real number σ defined here corresponds to a more general construction – in particular, it is the *Lebesgue outer measure* of the set X.) It can be shown (using the notion of *compactness*) that $(0,1)\subset X$ implies $\sigma \geq 1$ (since we did not discuss this notion in the course, we will not go into further detail – see Chapter 5 in [1] for treatment of this material in the context of several topics we talked about in the course).

On the other hand, the collection $(I_\ell)_{\ell\geq 1}$ is an admissible collection of intervals for the infimum used to define σ; we therefore obtain

$$\sigma \leq \sum_{n=1}^{\infty} 2^{-(n+1)} + \sum_{m=1}^{\infty} b_m - a_m \leq \frac{1}{2} + \frac{1}{4} < 1.$$

We have therefore shown both $\sigma \geq 1$ and $\sigma < 1$, which gives the desired contradiction. [1]

References

[1] G. Edgar. Measure, Topology, and Fractal Geometry. Springer UTM 2008, 2nd ed.

[1] We remark that the tools used here are particular instances of more general mathematical machinery – in particular, they are special cases of (i) the notion of *compactness* (which allows one to reduce the analysis of certain countable collections of open sets to the analysis of a finite subcollection), and (ii) properties of the outer measure.