18.S097 Introduction to Proofs
IAP 2015
Solution for Homework 4

Problem 1. *Define the function* $f : \mathbb{R} \to \mathbb{R}$ *by*

$$f(x) = x \quad \textit{for} \;\; x \in \mathbb{Q}$$

and

$$f(x) = x^2 \quad \textit{for} \;\; x \in \mathbb{R} \setminus \mathbb{Q}.$$

Show that f *is continuous at* 1 *and is not continuous at* 2.

Solution:

We first show that f is continuous at 1. Let $\epsilon > 0$ be given. We want to find $\delta > 0$ such that for all $x \in \mathbb{R}$, $|x - 1| < \delta$ implies $|f(x) - f(1)| = |f(x) - 1| < \epsilon$. Choose $\delta = \min\{1, \epsilon/3\}$ and let $x \in \mathbb{R}$ be given such that $|x - 1| < \delta$. We now observe that if $x \in \mathbb{Q}$ holds, then we have

$$|f(x) - 1| = |x - 1| < \delta \leq \frac{\epsilon}{3} < \epsilon,$$

while if $x \in \mathbb{R} \setminus \mathbb{Q}$, we have

$$|f(x) - 1| = |x^2 - 1| = |x + 1| \cdot |x - 1| \leq (|x - 1| + 2) \cdot |x - 1| < \delta^2 + 2\delta \leq 3\delta \leq \epsilon.$$

Thus, in either case, we have $|f(x) - 1| < \epsilon$. Since $\epsilon > 0$ was arbitrary, we have shown that f is continuous at 1.

We now show that f is not continuous at 2. As stated in the hint, it suffices to find $\epsilon > 0$ such that, for all for all $\delta > 0$, there exists $x \in \mathbb{R}$ with $|x - 2| < \delta$ and $|f(x) - f(2)| = |f(x) - 2| > \epsilon$. Set $\epsilon = 1$, and let $\delta > 0$ be given. We may then choose $x \in (2, 2 + \delta) \cap (\mathbb{R} \setminus \mathbb{Q})$ (e.g. by choosing $x = 2 + \sqrt{2}/n$ with n sufficiently large), and note that

$$|f(x) - 2| = |x^2 - 2| = |x + \sqrt{2}| \cdot |x - \sqrt{2}| \geq (2 + \sqrt{2})(2 - \sqrt{2}) = 2 > \epsilon.$$

This establishes the desired claim.