18.S097 Introduction to Proofs
IAP 2015
Homework 3
Due: Monday, Jan. 12, 2015

Problem 1. *Let A and B be arbitrarily given sets.*

(1) *Show that a function $f : A \to B$ is injective if and only if there exists a left inverse $g : B \to A$ for f in the sense that $g(f(x)) = x$ for all $x \in A$.*

Similarly, given two functions $f : A \to B$ and $g : B \to A$, we say that g is a right inverse for f if $f(g(x)) = x$ for all $x \in B$.

Establish the following claims:

(2) *A given function $f : A \to B$ is surjective if and only if there exists a right inverse $g : B \to A$ for f.*
(3) *Suppose that $f : A \to B$ has both a left inverse $g_1 : B \to A$ and a right inverse $g_2 : B \to A$. Then $g_1 = g_2$ (that is, $g_1(x) = g_2(x)$ for every $x \in B$).*

Solution:

(1): Let A, B be as stated and let $f : A \to B$ be a given function. We first show that f is injective if and only if there exists a left inverse $g : B \to A$. Suppose first that f is injective. Choose an element $x_0 \in A$. Note that since f is injective, the set $f^{-1}(\{b\})$ consists of exactly one element for each $b \in B$. We therefore define $g : B \to A$ as follows: if $b \in f(A)$, let $g(b)$ be the single element of $f^{-1}(\{b\})$, while if $b \notin f(A)$, set $g(b) = x_0$. Noting that for all $x \in A$, we have $f(x) \in f(A)$ (and moreover that x is the single element of $f^{-1}(\{f(x)\})$ in this case), we then have

$$g(f(x)) = x$$

for all $x \in A$. Thus, g is a left inverse as desired.

We now show the converse. Suppose that $g : B \to A$ is a left inverse for f. To show that f is injective, it suffices to show that for all $x, y \in A$, the condition $f(x) = f(y)$ implies $x = y$. We therefore let $x, y \in A$ be given such that $f(x) = f(y)$. Applying g to both sides and using the fact that g is a left inverse, we obtain

$$x = g(f(x)) = g(f(y)) = y,$$

as desired. Since x and y were arbitrary, we have shown that f is injective.

(2): We again let $f : A \to B$ be given. Suppose first that f is surjective. Then for every $b \in B$, we have $b \in f(A)$, so that we can find $x_b \in A$ such that $f(x_b) = b$. We now define $g : B \to A$ by setting $g(b) = x_b$ for every $b \in B$. We claim that g is a right inverse for f.

Let $y \in B$ be given. Then, with the correspondance $b \mapsto x_b$ chosen as above, we have $g(y) = x_y$, so that $f(g(y)) = f(x_y) = y$. Since $y \in B$ was arbitrary, g is a right inverse for f as desired.

It remains to show that the existence of a right inverse for f implies that f is surjective. Suppose that $g : B \to A$ is a right inverse, and let $b \in B$ be given. We want to find $a \in A$ such that $f(a) = b$. To do this, we set $a = g(b)$ and note that the right inverse condition implies $f(a) = f(g(b)) = b$ as desired. Thus f is surjective.

(3): Suppose that f, g_1 and g_2 are as stated, and let $x \in B$ be given. We then have (since $x = f(g_2(x))$)

$$g_1(x) = g_1\Big(f(g_2(x))\Big) = g_1\Big(f\Big(g_2(x)\Big)\Big) = g_2(x),$$

which was the desired result.