Introduction to Proofs
IAP 2015
Solution to Homework 1

Problem 1. *Let X be a set, and let $A, B \subset X$ be two subsets. Write down a rigorous proof of the equalities*

$$(A \cup B)^c = (A^c) \cap (B^c) \tag{1}$$

and

$$(A \cap B)^c = (A^c) \cup (B^c). \tag{2}$$

Hint: Use the style of proof which we saw on the example sheet. Each set equality consists of two inclusions.

To give a starting point, the argument for the inclusion $(A \cup B)^c \subset (A^c) \cap (B^c)$ might begin with: "Let $x \in (A \cup B)^c$ be given. We then have $x \in X$, but x does not belong to the set $A \cup B$ (that is, the statement "$x \in A$ or $x \in B$" is false). (...)"

Proof. We first show (1), for which we will establish the two inclusions $(A \cup B)^c \subset (A^c) \cap (B^c)$ and $(A^c) \cap (B^c) \subset (A \cup B)^c$ individually.

We begin with the first inclusion. Let $x \in (A \cup B)^c$ be given. We then have $x \in X$, but x does not belong to the set $A \cup B$ (that is, the statement "$x \in A$ or $x \in B$" is false). It follows that $x \notin A$ and $x \notin B$ both hold. Since these conditions can be rewritten as $x \in A^c$ and $x \in B^c$, respectively, we conclude x belongs to the set $(A^c) \cap (B^c)$ as desired. Since $x \in (A \cup B)^c$ was arbitrary, this implies the desired inclusion $(A \cup B)^c \subset (A^c) \cap (B^c)$.

We now show the second inclusion, $(A^c) \cap (B^c) \subset (A \cup B)^c$. Let $x \in (A^c) \cap (B^c)$ be given. We then have $x \in A^c$ and $x \in B^c$. Rephrasing, this means that x is an element of the set X for which the conditions $x \notin A$ and $x \notin B$ are both true. It follows that $x \in A \cup B$ is false (suppose for contradiction that $x \in A \cup B$; then $x \in A$ or $x \in B$, contradicting that we have both $x \notin A$ and $x \notin B$). Thus (since $x \in X$ by hypothesis), we have shown $x \in (A \cup B)^c$. Since $x \in (A^c) \cap (B^c)$ was arbitrary, the desired inclusion holds.

This completes the proof of (1). To show (2), note that it suffices to show

$$(C \cap D)^c = (C^c) \cup (D^c) \tag{3}$$

for all subsets $C, D \subset X$ (this is just the original claim restated with the names of A and B changed). For this, we will use (1) (which we have just shown to be valid), together with the fact that

$$(U^c)^c = U \tag{4}$$

holds for all sets $U \subset X$.[1]

[1]To verify (4), let $U \subset X$ be given; we will show $(U^c)^c \subset U$ and $U \subset (U^c)^c$. For the first inclusion, let $x \in (U^c)^c$ be given; then $x \in X$ and $x \notin U^c$. From these two conditions and the definition $U^c = \{x \in X : x \notin U\}$, it follows that $x \notin U$ is false – that is, x must belong to the set U. Since $x \in (U^c)^c$ was arbitrary, this gives the first inclusion.

Conversely, let $x \in U$ be given, and suppose for contradiction that $x \in U^c$. By the definition of U^c, we have $x \notin U$; this contradicts $x \in U$, and we conclude that $x \in U^c$ is false. Since $x \in U \subset X$, we therefore have $x \in (U^c)^c$, and we have shown the desired inclusion.

In particular, let $C, D \subset X$ be given and note that (4) gives

$$(C \cap D)^c = ((C^c)^c \cap (D^c)^c)^c.$$

An application of the first equality in the problem (with $A = C^c$ and $B = D^c$) now gives

$$(C \cap D)^c = \Big((C^c) \cup (D^c)\Big)^{cc}.$$

Using (4), we now see that the right hand side of this expression is equal to $(C^c) \cup (D^c)$, and the equality (3) holds as desired. $\square$