18.S097 Introduction to Proofs
IAP 2015
Example Sheet 1

Example 1. Show that "$p \Rightarrow q$" is equivalent to "not$(q) \Rightarrow$ not(p)" for any two propositions p and q.

Hint: Recall that the validity of $p \Rightarrow q$ is defined by the validity of the statement "not(p) or q".

Proof. Let p and q be given propositions. Suppose first that $p \Rightarrow q$ holds, i.e. that the statement "not(p) or q" is true. We want to show that

$$\text{``}\operatorname{not}(\operatorname{not}(q)) \text{ or } \operatorname{not}(p)'' \tag{1}$$

is true, which, by definition, is equivalent to not$(q) \Rightarrow$ not(p) (thus implying the first half of the desired equivalence). The validity of the statement (1) follows by observing that, for every proposition r, the proposition "not(not(r))" is equivalent to r, and that, for every proposition r and s, the propositions "r or s" and "s or r" are equivalent (each of these equivalences can be verified by examining the relevant truth tables).

Conversely, suppose that not$(q) \Rightarrow$ not(p) holds. By definition, this means that either not(not(q)) or not(p) are true. As in the previous argument, this pair of conditions is equivalent to q or not(p), which is equivalent to $p \Rightarrow q$, as desired. □

Example 2. Let X be a given set and let A, B be two arbitrary subsets of X. Show that

$$A \cap B \subset A.$$

Proof. Let $x \in A \cap B$ be given. By the definition of the intersection of sets, we then have the statement "$x \in A$ and $x \in B$" (so that in particular $x \in A$ holds). Since $x \in A \cap B$ was arbitrary, this shows the desired set inclusion. □

Example 3. Let X, A and B be as above. Show that

$$(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

Proof. We show the desired equality in two steps: by showing (i) $(A \setminus B) \cup (B \setminus A) \subset (A \cup B) \setminus (A \cap B)$, and (ii) $(A \cup B) \setminus (A \cap B) \subset (A \setminus B) \cup (B \setminus A)$.

(i): Let $x \in (A \setminus B) \cup (B \setminus A)$ be given. Then either $x \in A \setminus B$ or $x \in B \setminus A$. Suppose first that $x \in A \setminus B$. We then have $x \in A \subset A \cup B$. In order to show $x \in (A \cup B) \setminus (A \cap B)$, it remains to show that $x \notin A \cap B$. Suppose for contradiction that $x \in A \cap B$ holds. This in particular implies $x \in B$, contradicting $x \in A \setminus B$ (which holds by assumption). It follows that our original assumption $x \in A \cap B$ is false, so that $x \notin A \cap B$ holds as desired. We have therefore shown $x \in (A \cup B) \setminus (A \cap B)$ in this case.

Alternatively, suppose that $x \in B \setminus A$, so that $x \in B$ and $x \notin A$. It now follows from $B \subset A \cup B$ that $x \in A \cup B$, while $x \notin A$ implies $x \notin A \cap B$ (note that $A \cap B \subset A$, so that if $x \in A \cap B$ held, we would have $x \in A$)[1]. Thus $x \in (A \cup B) \setminus (A \cap B)$ holds in this case as well.

[1] This is really just a rephrasing of the argument in the previous paragraph.

Since x was arbitrary, we have shown the desired inclusion.

(ii): Let $x \in (A \cup B) \setminus (A \cap B)$ be given. We then have $x \in A \cup B$ and $x \notin A \cap B$. We split into two cases:

Case 1: $x \in A$.

In this case, $x \notin A \cap B$ implies $x \notin B$ (this is the contrapositive of the fact that for $x \in A$ one has the implication $x \in B$ implies $x \in A \cap B$). We therefore have $x \in A \setminus B \subset (A \setminus B) \cup (B \setminus A)$.

Case 2: $x \notin A$.

In this case, the condition $x \in A \cup B$ implies $x \in B$. On the other hand, $x \in B$ and $x \notin A \cap B$ together imply $x \notin A$ (otherwise, we would have $\square$

Remark 1. Some duplication in the previous proof can be avoided by noticing that the two cases in each of (i) and (ii) are symmetric in A and B. In particular, the proof of the second case in each of (i) and (ii) can be obtained from the first case by interchanging the roles of A and B. This can be made precise by formulating an intermediary claim; for instance note that in (i) both cases in the proof can be handled by showing the claim: For every $C, D \subset X$, one has $C \setminus D \subset (C \cup D) \setminus (C \cap D)$ (which can be established by proceeding as in the argument for either case), and taking $(C, D) = (A, B)$ followed by $(C, D) = (B, A)$.