## 1. Classification of 1-manifolds

**Theorem 1.1.** *Let $M$ be a connected 1-manifold. Then $M$ is diffeomorphic to $S^1$, $(0,1)$, $[0,1)$, or $[0,1]$.*

We start with a lemma:

**Lemma 1.2.** *Let $M$ be a 1-manifold, and let $f, g : (0,1) \to M$ be two parametrizations. Then $f(0,1) \cap g(0,1)$ has at most two components. If it has one component, then there is a parametrization $h : (0,1) \to M$ so that $h(0,1) = f(0,1) \cup g(0,1)$. If it has two components then $f(0,1) \cup g(0,1)$ is diffeomorphic to $S^1$.*

*Proof:* (Since this proof uses ordered pair and open intervals abundantly, we will use the notation $a \times b \in A \times B$ for the former to avoid confusion.) Let $\Gamma = \{s \times t; f(s) = g(t)\} \subseteq (0,1) \times (0,1)$. $\Gamma$ is a closed set, being the preimage of the diagonal $\Delta \subseteq M \times M$ by the map $f \times g : (0,1) \times (0,1) \to M \times M$. It can be thought of as the graph of the function $g^{-1} \circ f : f^{-1}(g(0,1)) \to (0,1)$, and also as the graph of $f^{-1} \circ g : g^{-1}(f(0,1)) \to (0,1)$.

Let $s_0 \in [0,1]$ be a point contained in the point-set boundary of $f^{-1}(g(0,1)) \subseteq (0,1) \subseteq [0,1]$. Then $s_0 \notin f^{-1}(g(0,1))$ since this set is open. If $s_k$ is a sequence in $f^{-1}(g(0,1))$ converging to $s_0$, consider the sequence $s_k \times (g^{-1} \circ f)(s_k) \in \Gamma$. Since $\Gamma$ is closed this sequence cannot converge in $(0,1) \times (0,1)$, but it must converge in $[0,1] \times [0,1]$ since it is a Cauchy sequence in a complete metric space. Therefore there are four possibilities: either $s_0 = 0$ or $s_0 = 1$, or $t_0 = \lim(g^{-1} \circ f)(s_k)$ is equal to either 0 or 1.

Since $\Gamma$ is graphical in both directions, it follows that distinct points $s_0$ in $\partial(f^{-1}(g(0,1))) \subseteq [0,1]$ must correspond to distinct cases above. Therefore the set $\partial(f^{-1}(g(0,1)))$ has at most four points, and since $f^{-1}(g(0,1))$ is an open subset of $(0,1)$ it follows that it has at most two connected components. This implies the first claim.

First, suppose that $f^{-1}(g(0,1))$ is connected. Then $\overline{\Gamma} \setminus \Gamma \subseteq [0,1] \times [0,1]$ consists of two points, both lying in $\partial([0,1] \times [0,1]) = \partial[0,1] \times [0,1] \cup [0,1] \times \partial[0,1]$. If both points lie in $\partial[0,1] \times [0,1]$ (or $[0,1] \times \partial[0,1]$) then $f^{-1}(g(0,1)) = (0,1)$ (respectively $g^{-1}(f(0,1)) = (0,1)$). Therefore $f(0,1) \subseteq g(0,1)$ $(\ldots g(0,1) \subseteq f(0,1))$ so we can take $h = g$ $(h = f)$. So the only interesting case is when one point is in $\partial[0,1] \times [0,1]$ and the other is in $[0,1] \times \partial[0,1]$.

By pre-composing either $f$ or $g$ with the diffeomorphism $(x \mapsto 1-x) : (0,1) \to (0,1)$, we can assume that $\overline{\Gamma} \setminus \Gamma = \{s_0 \times 0, 1 \times t_0\}$. So $g^{-1} \circ f : (s_0, 1) \to (0, t_0)$ is a diffeomorphism, $(g^{-1} \circ f)' > 0$. Restrict to smaller subintervals $(a,b) \subseteq (s_0, 1)$ $(c,d) \subseteq (0, t_0)$ so that $g^{-1} \circ f : (a,b) \to (c,d)$ is a diffeomorphism and $0 < \inf_{(a,b)}(g^{-1} \circ f)' < \sup_{(a,b)}(g^{-1} \circ f)' < \infty$.

Let $I_0 \subseteq I_1$, $J_0 \subseteq J_1$ be open intervals. We claim that, given any diffeomorphism $\varphi : I_0 \to J_0$ satisfying $0 < \inf \varphi' < \sup \varphi' < \infty$, there is a diffeomorphism $\Phi : I_1 \to J_1$ so that $\Phi|_{I_0} = \varphi$. Apply this to $g^{-1} \circ f$ to get a diffeomorphism $\alpha : (a, \frac{3}{2}) \to (c, 1)$. Then define $h : (0, \frac{3}{2}) \to M$ by

$$h(x) = \begin{cases} f(x) & x \in (0,b) \\ (g \circ \alpha)(x) & x \in (a, \frac{3}{2}) \end{cases}$$

which is well defined since $\alpha|_{(a,b)} = g^{-1} \circ f$. After rescaling the domain of $h$ the lemma is complete. ($h$ is obviously a local diffeomorphism. Check that the image of $h$ is the union of the images of $f$ and $g$! Also, why is $h$ injective?)

Now consider the case where $f^{-1}(g(0,1))$ has two components, so $\overline{\Gamma} \setminus \Gamma = \{s_0 \times 0, s_1 \times 1, 0 \times t_0, 1 \times t_1\}$. $\Gamma$ clearly cannot connect $s_0 \times 0$ to $s_1 \times 1$, otherwise $\Gamma$ would be necessarily connected since it is graphical. Therefore $\Gamma$ connects $s_0 \times 0$ to $1 \times t_1$ and $s_1 \times 1$ to $0 \times t_0$ (after possibly replacing $f(s)$ by $f(1-s)$). Since $g^{-1} \circ f$ is a diffeomorphism it is monotonic, and by the location of the endpoints we see that the only possibility is that $(g^{-1} \circ f)' > 0$ everywhere. This diffeomorphism restricts to two pieces: $\varphi_1 : (s_0, 1) \to (0, t_1)$ and $\varphi_2 : (0, s_1) \to (t_0, 1)$. Under the assumption that $\varphi_1$ and $\varphi_2$ have bounded derivatives, we claim there is a diffeomorphism $\alpha : (s_0, s_1 + 2\pi) \to (0, 1)$ so that $\alpha|_{(s_0,1)} = \varphi_1$ and $\alpha|_{(2\pi, s_1+2\pi)}(s) = \varphi_2(s - 2\pi)$. If they do not have bounded derivatives, we can choose smaller subintervals, as in the previous case.

Now, define $h : S^1 \to M$ by

$$h(\theta) = \begin{cases} f(\theta) & \theta \in (0,1) \\ (g \circ \alpha)(\theta) & \theta \in (s_0, s_1 + 2\pi) \end{cases}.$$

Notice that $h$ is well defined on $(s_0, 1)$ since $\alpha = g^{-1} \circ f$, also for $s > 0$ $\alpha(s) = (g^{-1} \circ f)(s - 2\pi)$ so $h$ is well defined on $S^1$. As before it's easy to check that $h$ is injective and therefore a diffeomorphism onto its image. $\square$