CHAPTER 4

FORMS ON MANIFOLDS

4.1 Manifolds

Our agenda in this chapter is to extend to manifolds the results of Chapters 2 and 3 and to formulate and prove manifold versions of two of the fundamental theorems of integral calculus: Stokes' theorem and the divergence theorem. In this section we'll define what we mean by the term "manifold", however, before we do so, a word of encouragement. Having had a course in multivariable calculus, you are already familiar with manifolds, at least in their one and two dimensional emanations, as curves and surfaces in $\mathbb{R}^3$, i.e., a manifold is basically just an n-dimensional surface in some high dimensional Euclidean space. To make this definition precise let X be a subset of $\mathbb{R}^N$, Y a subset of $\mathbb{R}^n$ and $f : X \to Y$ a continuous map. We recall

Definition 4.1.1. *f is a $\mathcal{C}^\infty$ map if for every $p \in X$, there exists a neighborhood, U_p, of p in $\mathbb{R}^N$ and a $\mathcal{C}^\infty$ map, $g_p : U_p \to \mathbb{R}^n$, which coincides with f on $U_p \cap X$.*

We also recall:

Theorem 4.1.2. *If $f : X \to Y$ is a $\mathcal{C}^\infty$ map, there exists a neighborhood, U, of X in $\mathbb{R}^N$ and a $\mathcal{C}^\infty$ map, $g : U \to \mathbb{R}^n$ such that g coincides with f on X.*

(A proof of this can be found in Appendix A.)

We will say that f is a *diffeomorphism* if it is one–one and onto and f and f^{-1} are both $\mathcal{C}^\infty$ maps. In particular if Y is an open subset of $\mathbb{R}^n$, X is an example of an object which we will call a *manifold*. More generally,

Definition 4.1.3. *A subset, X, of $\mathbb{R}^N$ is an n-dimensional manifold if, for every $p \in X$, there exists a neighborhood, V, of p in $\mathbb{R}^m$, an open subset, U, in $\mathbb{R}^n$, and a diffeomorphism $\varphi : U \to X \cap V$.*

Thus X is an n-dimensional manifold if, locally near every point p, X "looks like" an open subset of $\mathbb{R}^n$.

Some examples:

1. *Graphs of functions.* Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ a $\mathcal{C}^\infty$ function. Its graph

$$\Gamma_f = \{(x,t) \in \mathbb{R}^{n+1} ; \quad x \in U \, , \, t = f(x)\}$$

is an n-dimensional manifold in $\mathbb{R}^{n+1}$. In fact the map

$$\varphi : U \to \mathbb{R}^{n+1} \, , \quad x \to (x, f(x))$$

is a diffeomorphism of U onto Γ_f. (It's clear that φ is a $\mathcal{C}^\infty$ map, and it is a diffeomorphism since its inverse is the map, $\pi : \Gamma_f \to U$, $\pi(x,t) = x$, which is also clearly $\mathcal{C}^\infty$.)

2. *Graphs of mappings.* More generally if $f : U \to \mathbb{R}^k$ is a $\mathcal{C}^\infty$ map, its graph

$$\Gamma_f = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \, , \quad x \in U \, , \, y = f(x)\}$$

is an n-dimensional manifold in $\mathbb{R}^{n+k}$.

3. *Vector spaces.* Let V be an n- dimensional vector subspace of $\mathbb{R}^N$, and $(e_1, \ldots, e_n)$ a basis of V. Then the linear map

$$\varphi : \mathbb{R}^n \to V \, , \quad (x_1, \ldots, x_n) \to \sum x_i e_i \tag{4.1.1}$$

is a diffeomorphism of $\mathbb{R}^n$ onto V. Hence every n-dimensional vector subspace of $\mathbb{R}^N$ is automatically an n-dimensional submanifold of $\mathbb{R}^N$. Note, by the way, that if V is *any* n-dimensional vector space, not necessarily a subspace of $\mathbb{R}^N$, the map (5.1.1) gives us an identification of V with $\mathbb{R}^n$. This means that we can speak of subsets of V as being k-dimensional submanifolds if, via this identification, they get mapped onto k-dimensional submanifolds of $\mathbb{R}^n$. (This is a trivial, but useful, observation since a lot of interesting manifolds occur "in nature" as subsets of some abstract vector space rather than explicitly as subsets of some $\mathbb{R}^n$. An example is the manifold, $O(n)$, of orthogonal $n \times n$ matrices. (See example 10 below.) This manifold occurs in nature as a submanifold of the vector space of n by n matrices.)

4. *Affine subspaces* of $\mathbb{R}^n$. These are manifolds of the form $p + V$, where V is a vector subspace of $\mathbb{R}^N$, and p is some specified point in

$\mathbb{R}^N$. In other words, they are diffeomorphic copies of the manifolds in example 3 with respect to the diffeomorphism

$$\tau_p : \mathbb{R}^N \times \mathbb{R}^N , \quad x \to x + p .$$

If X is an arbitrary submanifold of $\mathbb{R}^N$ its *tangent space* a point, $p \in X$, is an example of a manifold of this type. (We'll have more to say about tangent spaces in §4.2.)

5. *Product manifolds.* Let X_i, $i = 1, 2$ be an n_i-dimensional submanifold of $\mathbb{R}^{N_i}$. Then the Cartesian product of X_1 and X_2

$$X_1 \times X_2 = \{(x_1, x_2) \,;\, x_i \in X_i\}$$

is an n-dimensional submanifold of $\mathbb{R}^N$ where $n = n_1 + n_2$ and $\mathbb{R}^N = \mathbb{R}^{N_1} \to \mathbb{R}^{N_2}$.

We will leave for you to verify this fact as an exercise. *Hint:* For $p_i \in X_i$, $i = 1, 2$, there exists a neighborhood, V_i, of p_i in $\mathbb{R}^{N_i}$, an open set, U_i in $\mathbb{R}^{n_i}$, and a diffeomorphism $\varphi : U_i \to X_i \cap V_i$. Let $U = U_1 \times U_2$, $V = V_1 \times V_2$ and $X = X_1 \times X_2$, and let $\varphi : U \to X \cap V$ be the product diffeomorphism, $(\varphi(q_1), \varphi_2(q_2))$.

6. *The unit n-sphere.* This is the set of unit vectors in $\mathbb{R}^{n+1}$:

$$S^n = \{x \in \mathbb{R}^{n+1} , \quad x_1^2 + \cdots + x_{n+1}^2 = 1\} .$$

To show that S^n is an n-dimensional manifold, let V be the open subset of $\mathbb{R}^{n+1}$ on which x_{n+1} is positive. If U is the open unit ball in $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ is the function, $f(x) = (1 - (x_1^2 + \cdots + x_n^2))^{1/2}$, then $S^n \cap V$ is just the graph, Γ_f, of f as in example 1. So, just as in example 1, one has a diffeomorphism

$$\varphi : U \to S^n \cap V .$$

More generally, if $p = (x_1, \ldots, x_{n+1})$ is any point on the unit sphere, then x_i is non-zero for some i. If x_i is positive, then letting σ be the transposition, $i \leftrightarrow n + 1$ and $f_\sigma : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, the map

$$f_\sigma(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

one gets a diffeomorphism, $f_\sigma \circ \varphi$, of U onto a neighborhood of p in S^n and if x_i is negative one gets such a diffeomorphism by replacing f_σ by $-f_\sigma$. In either case we've shown that for every point, p, in S^n, there is a neighborhood of p in S^n which is diffeomorphic to U.

7. *The* 2*-torus.* In calculus books this is usually described as the surface of rotation in $\mathbb{R}^3$ obtained by taking the unit circle centered at the point, $(2,0)$, in the (x_1,x_3) plane and rotating it about the x_3-axis. However, a slightly nicer description of it is as the product manifold $S^1 \times S^1$ in $\mathbb{R}^4$. (*Exercise:* Reconcile these two descriptions.)

We'll now turn to an alternative way of looking at manifolds: as *solutions of systems of equations*. Let U be an open subset of $\mathbb{R}^N$ and $f : U \to \mathbb{R}^k$ a $\mathcal{C}^\infty$ map.

Definition 4.1.4. *A point, $a \in \mathbb{R}^k$, is a regular value of f if for every point, $p \in f^{-1}(a)$, f is a submersion at p.*

Note that for f to be a submersion at p, $Df(p) : \mathbb{R}^N \to \mathbb{R}^k$ has to be onto, and hence k has to be less than or equal to N. Therefore this notion of "regular value" is interesting only if $N \geq k$.

Theorem 4.1.5. *Let $N - k = n$. If a is a regular value of f, the set, $X = f^{-1}(a)$, is an n-dimensional manifold.*

Proof. Replacing f by $\tau_{-a} \circ f$ we can assume without loss of generality that $a = 0$. Let $p \in f^{-1}(0)$. Since f is a submersion at p, the canonical submersion theorem (see Appendix B, Theorem 2) tells us that there exists a neighborhood, $\mathcal{O}$, of 0 in $\mathbb{R}^N$, a neighborhood, U_0, of p in U and a diffeomorphism, $g : \mathcal{O} \to U_0$ such that

$$f \circ g = \pi \tag{4.1.2}$$

where π is the projection map

$$\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^k\,, \quad (x,y) \to x\,.$$

Hence $\pi^{-1}(0) = \{0\} \times \mathbb{R}^n = \mathbb{R}^n$ and by (5.1.1), g maps $\mathcal{O} \cap \pi^{-1}(0)$ diffeomorphically onto $U_0 \cap f^{-1}(0)$. However, $\mathcal{O} \cap \pi^{-1}(0)$ is a neighborhood, V, of 0 in $\mathbb{R}^n$ and $U_0 \cap f^{-1}(0)$ is a neighborhood of p in X, and, as remarked, these two neighborhoods are diffeomorphic.

□

Some examples:

8. *The n-sphere.* Let

$$f : \mathbb{R}^{n+1} \to \mathbb{R}$$

be the map,

$$(x_1, \dots, x_{n+1}) \to x_1^2 + \cdots + x_{n+1}^2 - 1\,.$$

Then

$$Df(x) = 2(x_1, \dots, x_{n+1})$$

so, if $x \neq 0$ f is a submersion at x. In particular f is a submersion at all points, x, on the n-sphere

$$S^n = f^{-1}(0)$$

so the n-sphere is an n-dimensional submanifold of $\mathbb{R}^{n+1}$.

9. *Graphs.* Let $g : \mathbb{R}^n \to \mathbb{R}^k$ be a $\mathcal{C}^\infty$ map and as in example 2 let

$$\Gamma_f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k\,, \quad y = g(x)\}\,.$$

We claim that Γ_f is an n-dimensional submanifold of $\mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$.

Proof. Let

$$f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$$

be the map, $f(x, y) = y - g(x)$. Then

$$Df(x, y) = [-Dg(x)\,,\ I_k]$$

where I_k is the identity map of $\mathbb{R}^k$ onto itself. This map is always of rank k. Hence $\Gamma_f = f^{-1}(0)$ is an n-dimensional submanifold of R^{n+k}.

□

10. Let $\mathcal{M}_n$ be the set of all $n \times n$ matrices and let $\mathcal{S}_n$ be the set of all symmetric $n \times n$ matrices, i.e., the set

$$\mathcal{S}_n = \{A \in \mathcal{M}_n\,,\ A = A^t\}\,.$$

The map

$$[a_{i,j}] \to (a_{11}, a_{12}, \dots, a_{1n}, a_{2,1}, \dots, a_{2n}, \dots)$$

gives us an identification

$$\mathcal{M}_n \cong \mathbb{R}^{n^2}$$

and the map

$$[a_{i,j}] \to (a_{11}, \dots a_{1n}, a_{22}, \dots a_{2n}, a_{33}, \dots a_{3n}, \dots)$$

gives us an identification

$$\mathcal{S}_n \cong \mathbb{R}^{\frac{n(n+1)}{2}} .$$

(Note that if A is a symmetric matrix,

$$a_{12} = a_{21} \, , \; a_{13} = a_{31} \, , \; a_{32} = a_{23}, \text{ etc.}$$

so this map avoids redundancies.) Let

$$O(n) = \{A \in \mathcal{M}_n \, , \; A^t A = I\} \, .$$

This is the set of *orthogonal* $n \times n$ matrices, and we will leave for you as an exercise to show that it's an $n(n-1)/2$-dimensional manifold.

Hint: Let $f : \mathcal{M}_n \to \mathcal{S}_n$ be the map $f(A) = A^t A - I$. Then

$$O(n) = f^{-1}(0) \, .$$

These examples show that lots of interesting manifolds arise as zero sets of submersions, $f : U \to \mathbb{R}^k$. This is, in fact, not just an accident. We will show that locally *every* manifold arises this way. More explicitly let $X \subseteq \mathbb{R}^N$ be an n-dimensional manifold, p a point of X, U a neighborhood of 0 in $\mathbb{R}^n$, V a neighborhood of p in $\mathbb{R}^N$ and $\varphi : (U, 0) \to (V \cap X, p)$ a diffeomorphism. We will for the moment think of φ as a $\mathcal{C}^\infty$ map $\varphi : U \to \mathbb{R}^N$ whose image happens to lie in X.

Lemma 4.1.6. *The linear map*

$$D\varphi(0) : \mathbb{R}^n \to \mathbb{R}^N$$

is injective.

Proof. $\varphi^{-1} : V \cap X \to U$ is a diffeomorphism, so, shrinking V if necessary, we can assume that there exists a $\mathcal{C}^\infty$ map $\psi : V \to U$ which coincides with φ^{-1}on $V \cap X$ Since φ maps U onto $V \cap X$, $\psi \circ \varphi = \varphi^{-1} \circ \varphi$ is the identity map on U. Therefore,

$$D(\psi \circ \varphi)(0) = (D\psi)(p) D\varphi(0) = I$$

by the chain rule, and hence if $D\varphi(0)v = 0$, it follows from this identity that $v = 0$.

$\square$

Lemma 5.1.6 says that φ is an immersion at 0, so by the canonical immersion theorem (see Appendix B,Theorem 4) there exists a neighborhood, U_0, of 0 in U, a neighborhood, V_p, of p in V, and a diffeomorphism

$$g : (V_p, p) \to (U_0 \times \mathbb{R}^{N-n}, 0) \tag{4.1.3}$$

such that

$$g \circ \varphi = \iota \,, \tag{4.1.4}$$

ι being, as in Appendix B, the canonical immersion

$$\iota : U_0 \to U_0 \times \mathbb{R}^{N-n} \,, \quad x \to (x, 0) \,. \tag{4.1.5}$$

By (5.1.3) g maps $\varphi(U_0)$ diffeomorphically onto $\iota(U_0)$. However, by (5.1.2) and (5.1.3) $\iota(U_0)$ is defined by the equations, $x_i = 0$, $i = n+1, \dots, N$. Hence if $g = (g_1, \dots, g_N)$ the set, $\varphi(U_0) = V_p \cap X$ is defined by the equations

$$g_i = 0 \,, \quad i = n+1, \dots, N \,. \tag{4.1.6}$$

Let $\ell = N - n$, let

$$\pi : \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^\ell \to \mathbb{R}^\ell$$

be the canonical submersion,

$$\pi(x_1, \dots, x_N) = (x_{n+1}, \dots x_N)$$

and let $f = \pi \circ g$. Since g is a diffeomorphism, f is a submersion and (5.1.5) can be interpreted as saying that

$$V_p \cap X = f^{-1}(0) \,. \tag{4.1.7}$$

Thus to summarize we've proved

Theorem 4.1.7. *Let X be an n-dimensional submanifold of $\mathbb{R}^N$ and let $\ell = N - n$. Then for every $p \in X$ there exists a neighborhood, V_p, of p in $\mathbb{R}^N$ and a submersion*

$$f : (V_p, p) \to (\mathbb{R}^\ell, 0)$$

such that $X \cap V_p$ is defined by the equation (5.1.6).

A nice way of thinking about Theorem 4.1.2 is in terms of the coordinates of the mapping, f. More specifically if $f = (f_1, \ldots, f_k)$ we can think of $f^{-1}(a)$ as being the set of solutions of the system of equations

$$f_i(x) = a_i\,, \quad i = 1, \ldots, k \tag{4.1.8}$$

and the condition that a be a regular value of f can be interpreted as saying that for every solution, p, of this system of equations the vectors

$$(df_i)_p = \sum \frac{\partial f_i}{\partial x_j}(0)\, dx_j \tag{4.1.9}$$

in $T_p^*\mathbb{R}^n$ are linearly independent, i.e., the system (5.1.7) is an "independent system of defining equations" for X.

Exercises.

1. Show that the set of solutions of the system of equations

$$x_1^2 + \cdots + x_n^2 \;=\; 1$$

and

$$x_1 + \cdots + x_n \;=\; 0$$

is an $n-2$-dimensional submanifold of $\mathbb{R}^n$.

2. Let S^{n-1} be the n-sphere in $\mathbb{R}^n$ and let

$$X_a = \{x \in S^{n-1}\,, \quad x_1 + \cdots + x_n = a\}\,.$$

For what values of a is X_a an $(n-2)$-dimensional submanifold of S^{n-1}?

3. Show that if X_i, $i = 1, 2$, is an n_i-dimensional submanifold of $\mathbb{R}^{N_i}$ then

$$X_1 \times X_2 \subseteq \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$$

is an $(n_1 + n_2)$-dimensional submanifold of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$.

4. Show that the set

$$X = \{(x, \mathrm{v}) \in S^{n-1} \times \mathbb{R}^n\,, \quad x \cdot \mathrm{v} = 0\}$$

is a $2n-2$-dimensional submanifold of $\mathbb{R}^n \times \mathbb{R}^n$. (Here "$x \cdot \mathrm{v}$" is the dot product, $\sum x_i v_i$.)

5. Let $g : \mathbb{R}^n \to \mathbb{R}^k$ be a $\mathcal{C}^\infty$ map and let $X = \operatorname{graph} g$. Prove directly that X is an n-dimensional manifold by proving that the map

$$\gamma : \mathbb{R}^n \to X\,, \qquad x \to (x, g(x))$$

is a diffeomorphism.

6. Prove that $O(n)$ is an $n(n-1)/2$-dimensional manifold. *Hints:*

(a) Let $f : \mathcal{M}_n \to \mathcal{S}_n$ be the map

$$f(A) = A^t A = I\,.$$

Show that $O(n) = f^{-1}(0)$.

(b) Show that

$$f(A + \epsilon B) = A^t A + \epsilon(A^t B + B^t A) + \epsilon^2 B^t B\,.$$

(c) Conclude that the derivative of f at A is the map

$$B \in \mathcal{M}_n \to A^t B + B^t A\,. \tag{*}$$

(d) Let A be in $O(n)$. Show that if C is in $\mathcal{S}_n$ and $B = AC/2$ then the map, (*), maps B onto C.

(e) Conclude that the derivative of f is surjective at A.

(f) Conclude that 0 is a regular value of the mapping, f.

7. The next five exercises, which are somewhat more demanding than the exercises above, are an introduction to "Grassmannian" geometry.

(a) Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$ and let $W = \operatorname{span}\{e_{k+1}, \ldots, e_n\}$. Prove that if V is a k-dimensional subspace of $\mathbb{R}^n$ and

$$V \cap W = \{0\}\,, \tag{1.1}$$

then one can find a *unique* basis of V of the form

$$\mathrm{v}_i = e_i + \sum_{j=1}^{\ell} b_{i,j} e_{k+j}\,, \quad i = 1, \dots, k\,, \tag{1.2}$$

where $\ell = n - k$.

(b) Let G_k be the set of k-dimensional subspaces of $\mathbb{R}^n$ having the property (1.1) and let $\mathcal{M}_{k,\ell}$ be the vector space of $k \times \ell$ matrices. Show that one gets from the identities (1.2) a bijective map:

$$\gamma : \mathcal{M}_{k,\ell} \to G_k\,. \tag{1.3}$$

8. Let S_n be the vector space of linear mappings of $\mathbb{R}^n$ into itself which are self-adjoint, i.e., have the property $A = A^t$.

(a) Given a k-dimensional subspace, V of $\mathbb{R}^n$ let $\pi_V : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection of $\mathbb{R}^n$ onto V. Show that π_V is in S_n and is of rank k, and show that $(\pi_V)^2 = \pi_V$.

(b) Conversely suppose A is an element of S_n which is of rank k and has the property, $A^2 = A$. Show that if V is the image of A in $\mathbb{R}^n$, then $A = \pi_V$.

Notation. We will call an $A \in S_n$ of the form, $A = \pi_V$ above a *rank k projection operator.*

9. Composing the map

$$\rho : G_k \to S_n\,, \quad V \to \pi_V \tag{1.4}$$

with the map (1.3) we get a map

$$\varphi : \mathcal{M}_{k,\ell} \to S_n\,, \quad \varphi = \rho \cdot \gamma\,. \tag{1.5}$$

Prove that φ is $\mathcal{C}^\infty$.

Hints:

(a) By Gram–Schmidt one can convert (1.2) into an orthonormal basis

$$e_{1,B}, \dots, e_{n,B} \tag{1.6}$$

of V. Show that the $e_{i,B}$'s are $\mathcal{C}^\infty$ functions of the matrix, $B = [b_{i,j}]$.

(b) Show that π_V is the linear mapping

$$\mathrm{v} \in V \to \sum_{i=1}^{k} (\mathrm{v} \cdot e_{i,B}) e_{i,B} \,.$$

10. Let $V_0 = \operatorname{span}\{e_1, \dots, e_n\}$ and let $\widetilde{G}_k = \rho(G_k)$. Show that φ maps a neighborhood of 0 in $\mathcal{M}_{k,\ell}$ diffeomorphically onto a neighborhood of π_{V_0} in $\widetilde{G}_k$.

Hints: π_V is in $\widetilde{G}_k$ if and only if V satisfies (1.1). For $1 \le i \le k$ let

$$w_i = \pi_V(e_i) = \sum_{j=1}^{k} a_{i,j} e_j + \sum_{r=1}^{\ell} c_{i,r} e_{k+r} \,. \tag{1.7}$$

(a) Show that if the matrix $A = [a_{i,j}]$ is invertible, π_V is in $\widetilde{G}_k$.

(b) Let $\mathcal{O} \subseteq \widetilde{G}_k$ be the set of all π_V's for which A is invertible. Show that $\varphi^{-1} : \mathcal{O} \to \mathcal{M}_{k,\ell}$ is the map

$$\varphi^{-1}(\pi_V) = B = A^{-1}C$$

where $C = [c_{i,j}]$.

11. Let $G(k,n) \subseteq S_n$ be the set of rank k projection operators. Prove that $G(k,n)$ is a $k\ell$-dimensional submanifold of the Euclidean space, $S_n = \mathbb{R}^{\frac{n(n+1)}{2}}$.

Hints:

(a) Show that if V is any k-dimensional subspace of $\mathbb{R}^n$ there exists a linear mapping, $A \in O(n)$ mapping V_0 to V.

(b) Show that $\pi_V = A\pi_{V_0}A^{-1}$.

(c) Let $K_A : S_n \to S_n$ be the linear mapping,

$$K_A(B) = ABA^{-1} \,.$$

Show that

$$K_A \cdot \varphi : \mathcal{M}_{k,\ell} \to S_n$$

maps a neighborhood of 0 in $\mathcal{M}_{k,\ell}$ diffeomorphically onto a neighborhood of π_V in $G(k,n)$.

Remark 4.1.8. *Let $Gr(k,n)$ be the set of all k-dimensional subspaces of $\mathbb{R}^n$. The identification of $Gr(k,n)$ with $G(k,n)$ given by $V \leftrightarrow \pi_V$ allows us to restate the result above in the form.*

The "Grassmannnian" Theorem: The set $(\mathrm{G}r(k,n))$ (a.k.a. the "Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$") is a $k\ell$-dimensional submanifold of $S_n = \mathbb{R}^{\frac{n(n+1)}{2}}$.

12. Show that $\mathrm{G}r(k,n)$ is a compact submanifold of S_n. *Hint:* Show that it's closed and bounded.

4.2 Tangent spaces

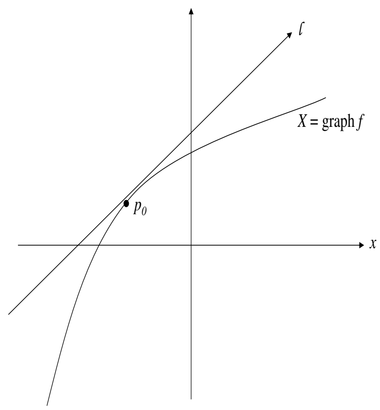
We recall that a subset, X, of $\mathbb{R}^N$ is an n-dimensional manifold, if, for every $p \in X$, there exists an open set, $U \subseteq \mathbb{R}^n$, a neighborhood, V, of p in $\mathbb{R}^N$ and a $\mathcal{C}^\infty$-diffeomorphism, $\varphi : U \to X \cap X$.

Definition 4.2.1. *We will call φ a* parametrization *of X at p.*

Our goal in this section is to define the notion of the *tangent space*, T_pX, to X at p and describe some of its properties. Before giving our official definition we'll discuss some simple examples.

Example 1.

Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function and let $X = \mathrm{graph} f$.



Then in this figure above the tangent line, ℓ, to X at $p_0 = (x_0, y_0)$ is defined by the equation

$$y - y_0 = a(x - x_0)$$

where $a = f'(x_0)$ In other words if p is a point on ℓ then $p = p_0 + \lambda \mathrm{v}_0$ where $\mathrm{v}_0 = (1, a)$ and $\lambda \in \mathbb{R}$. We would, however, like the tangent space to X at p_0 to be a subspace of the tangent space to $\mathbb{R}^2$ at p_0, i.e., to be the subspace of the space: $T_{p_0}\mathbb{R}^2 = \{p_0\} \times \mathbb{R}^2$, and this we'll achieve by defining

$$T_{p_0}X = \{(p_0, \lambda \mathrm{v}_0), \quad \lambda \in \mathbb{R}\}.$$

Example 2.

Let S^2 be the unit 2-sphere in $\mathbb{R}^3$. The tangent plane to S^2 at p_0 is usually defined to be the plane

$$\{p_0 + \mathrm{v}\,;\, v \in \mathbb{R}^3\,, \quad \mathrm{v} \perp p_0\}.$$

However, this tangent plane is easily converted into a subspace of $T_p\mathbb{R}^3$ via the map, $p_0 + \mathrm{v} \to (p_0, \mathrm{v})$ and the image of this map

$$\{(p_0, \mathrm{v})\,;\, \mathrm{v} \in \mathbb{R}^3\,, \quad \mathrm{v} \perp p_0\}$$

will be our definition of $T_{p_0}S^2$.

Let's now turn to the general definition. As above let X be an n-dimensional submanifold of $\mathbb{R}^N$, p a point of X, V a neighborhood of p in $\mathbb{R}^N$, U an open set in $\mathbb{R}^n$ and

$$\varphi : (U, q) \to (X \cap V, p)$$

a parameterization of X. We can think of φ as a $\mathcal{C}^\infty$ map

$$\varphi : (U, q) \to (V, p)$$

whose image happens to lie in $X \cap V$ and we proved in §5.1 that its derivative at q

$$(d\varphi)_q : T_q\mathbb{R}^n \to T_p\mathbb{R}^N \tag{4.2.1}$$

is injective.

Definition 4.2.2. *The tangent space, T_pX, to X at p is the image of the linear map (4.2.1). In other words, $w \in T_p\mathbb{R}^N$ is in T_pX if and only if $w = d\varphi_q(\mathrm{v})$ for some $\mathrm{v} \in T_q\mathbb{R}^n$. More succinctly,*

$$T_pX = (d\varphi_q)(T_q\mathbb{R}^n)\,. \tag{4.2.2}$$

(Since $d\varphi_q$ is injective this space is an n-dimensional vector subspace of $T_p\mathbb{R}^N$.)

One problem with this definition is that it appears to depend on the choice of φ. To get around this problem, we'll give an alternative definition of T_pX. In §5.1 we showed that there exists a neighborhood, V, of p in $\mathbb{R}^N$ (which we can without loss of generality take to be the same as V above) and a $\mathcal{C}^\infty$ map

$$f : (V, p) \to (\mathbb{R}^k, 0)\,, \quad k = N - n\,, \tag{4.2.3}$$

such that $X \cap V = f^{-1}(0)$ and such that f is a submersion at all points of $X \cap V$, and in particular at p. Thus

$$df_p : T_p\mathbb{R}^N \to T_0\mathbb{R}^k$$

is surjective, and hence the kernel of df_p has dimension n. Our alternative definition of T_pX is

$$T_pX = \mathrm{k}\mathit{ernel}\, df_p\,. \tag{4.2.4}$$

The spaces (4.2.2) and (4.2.4) are both n-dimensional subspaces of $T_p\mathbb{R}^N$, and we claim that these spaces are the same. (Notice that the definition (4.2.4) of T_pX doesn't depend on φ, so if we can show that these spaces are the same, the definitions (4.2.2) and (4.2.4) will depend *neither* on φ *nor* on f.)

Proof. Since $\varphi(U)$ is contained in $X \cap V$ and $X \cap V$ is contained in $f^{-1}(0)$, $f \circ \varphi = 0$, so by the chain rule

$$df_p \circ d\varphi_q = d(f \circ \varphi)_q = 0\,. \tag{4.2.5}$$

Hence if $\mathrm{v} \in T_p\mathbb{R}^n$ and $w = d\varphi_q(\mathrm{v})$, $df_p(w) = 0$. This shows that the space (4.2.2) is contained in the space (4.2.4). However, these two spaces are n-dimensional so they coincide.

□

From the proof above one can extract a slightly stronger result:

Theorem 4.2.3. *Let W be an open subset of $\mathbb{R}^\ell$ and $h : (W, q) \to (\mathbb{R}^N, p)$ a $\mathcal{C}^\infty$ map. Suppose $h(W)$ is contained in X. Then the image of the map*

$$dh_q : T_q\mathbb{R}^\ell \to T_p\mathbb{R}^N$$

is contained in T_pX.

Proof. Let f be the map (4.2.3). We can assume without loss of generality that $h(W)$ is contained in V, and so, by assumption, $h(W) \subseteq X \cap V$. Therefore, as above, $f \circ h = 0$, and hence $dh_q(T_q\mathbb{R}^\ell)$ is contained in the kernel of df_p.

□

This result will enable us to define the *derivative* of a mapping between manifolds. Explicitly: Let X be a submanifold of $\mathbb{R}^N$, Y a submanifold of $\mathbb{R}^m$ and $g : (X, p) \to (Y, y_0)$ a $\mathcal{C}^\infty$ map. By Definition 5.1.1 there exists a neighborhood, $\mathcal{O}$, of X in $\mathbb{R}^N$ and a $\mathcal{C}^\infty$ map, $\widetilde{g} : \mathcal{O} \to \mathbb{R}^m$ extending to g. We will define

$$(dg_p) : T_pX \to T_{y_0}Y \tag{4.2.6}$$

to be the restriction of the map

$$(d\widetilde{g})_p : T_p\mathbb{R}^N \to T_{y_0}\mathbb{R}^m \tag{4.2.7}$$

to T_pX. There are two obvious problems with this definition:

1. Is the space
$$(d\widetilde{g}_p)(T_pX)$$
contained in $T_{y_0}Y$?

2. Does the definition depend on $\widetilde{g}$?

To show that the answer to 1. is yes and the answer to 2. is no, let
$$\varphi : (U, x_0) \to (X \cap V, p)$$
be a parametrization of X, and let $h = \widetilde{g} \circ \varphi$. Since $\varphi(U) \subseteq X$, $h(U) \subseteq Y$ and hence by Theorem 4.2.4
$$dh_{x_0}(T_{x_0}\mathbb{R}^n) \subseteq T_{y_0}Y\,.$$
But by the chain rule
$$dh_{x_0} = d\widetilde{g}_p \circ d\varphi_{x_0}\,, \tag{4.2.8}$$
so by (4.2.2)
$$(d\widetilde{g}_p)(T_pX) \quad \subseteq \quad T_pY \tag{4.2.9}$$
and
$$(d\widetilde{g}_p)(T_pX) \quad = \quad (dh)_{x_0}(T_{x_0}\mathbb{R}^n) \tag{4.2.10}$$
Thus the answer to 1. is yes, and since $h = \widetilde{g} \circ \varphi = g \circ \varphi$, the answer to 2. is no.

From (4.2.5) and (4.2.6) one easily deduces

Theorem 4.2.4 (Chain rule for mappings between manifolds)**.** *Let Z be a submanifold of $\mathbb{R}^\ell$ and $\psi : (Y, y_0) \to (Z, z_0)$ a $\mathcal{C}^\infty$ map. Then $d\psi_{y_0} \circ dg_p = d(\psi \circ g)_p$.*

We will next prove manifold versions of the inverse function theorem and the canonical immersion and submersion theorems.

Theorem 4.2.5 (Inverse function theorem for manifolds)**.** *Let X and Y be n-dimensional manifolds and $f : X \to Y$ a $\mathcal{C}^\infty$ map. Suppose that at $p \in X$ the map*
$$df_p : T_pX \to T_qY\,, \quad q = f(p)\,,$$
is bijective. Then f maps a neighborhood, U, of p in X diffeomorphically onto a neighborhood, V, of q in Y.

Proof. Let U and V be open neighborhoods of p in X and q in Y and let

$$\varphi_0 : (U_0, p_0) \to (U, p)$$

and

$$\psi_0 : (V_0, q_0) \to (V, q)$$

be parametrizations of these neighborhoods. Shrinking U_0 and U we can assume that $f(U) \subseteq V$. Let

$$g : (U_0, p_0) \to (V_0, q_0)$$

be the map $\psi_0^{-1} \circ f \circ \varphi_0$. Then $\psi_0 \circ g = f \circ \varphi$, so by the chain rule

$$(d\psi_0)_{q_0} \circ (dg)_{p_0} = (df)_p \circ (d\varphi_0)_{p_0} .$$

Since $(d\psi_0)_{q_0}$ and $(d\varphi_0)_{p_0}$ are bijective it's clear from this identity that if df_p is bijective the same is true for $(dg)_{p_0}$. Hence by the inverse function theorem for open subsets of $\mathbb{R}^n$, g maps a neighborhood of p_0 in U_0 diffeomorphically onto a neighborhood of q_0 in V_0. Shrinking U_0 and V_0 we assume that these neighborhoods are U_0 and V_0 and hence that g is a diffeomorphism. Thus since $f : U \to V$ is the map $\psi_0 \circ g \circ \varphi_0^{-1}$, it is a diffeomorphism as well.

□

Theorem 4.2.6 (The canonical submersion theorem for manifolds)**.** *Let X and Y be manifolds of dimension n and m, $m < n$, and let $f : X \to Y$ be a $\mathcal{C}^\infty$ map. Suppose that at $p \in X$ the map*

$$df_p : T_pX \to T_qY \,, \quad q = f(p) \,,$$

is surjective. Then there exists an open neighborhood, U, of p in X, and open neighborhood, V of $f(U)$ in Y and parametrizations

$$\varphi_0 : (U_0, 0) \to (U, p)$$

and

$$\psi_0 : (V_0, 0) \to (V, q)$$

such that in the diagram below

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ {\scriptstyle\varphi_0}\big\uparrow & & \big\uparrow{\scriptstyle\psi_0} \\ U_0 & \longrightarrow & V_0 \end{array}$$

the bottom arrow, $\psi_0^{-1} \circ f \circ \varphi_0$*, is the canonical submersion,* π*.*

Proof. Let U and V be open neighborhoods of p and q and

$$\varphi_0 : (U_0, p_0) \to (U, p)$$

and

$$\psi_0 : (V_0, q_0) \to (V, q)$$

be parametrizations of these neighborhoods. Composing φ_0 and ψ_0 with the translations we can assume that p_0 is the origin in $\mathbb{R}^n$ and q_0 the origin in $\mathbb{R}^m$, and shrinking U we can assume $f(U) \subseteq V$. As above let $g : (U_0, 0) \to (V_0, 0)$ be the map, $\psi_0^{-1} \circ f \circ \varphi_0$. By the chain rule

$$(d\psi_0)_0 \circ (dg)_0 = df_p \circ (d\varphi_0)_0 \,,$$

therefore, since $(d\psi_0)_0$ and $(d\varphi_0)_0$ are bijective it follows that $(dg)_0$ is surjective. Hence, by Theorem 5.1.2, we can find an open neighborhood, U, of the origin in $\mathbb{R}^n$ and a diffeomorphism, $\varphi_1 : (U_1, 0) \to (U_0, 0)$ such that $g \circ \varphi_1$ is the canonical submersion. Now replace U_0 by U_1 and φ_0 by $\varphi_0 \circ \varphi_1$.

□

Theorem 4.2.7 (The canonical immersion theorem for manifolds)**.** *Let* X *and* Y *be manifolds of dimension* n *and* m*,* $n < m$*, and* $f : X \to Y$ *a* $\mathcal{C}^\infty$ *map. Suppose that at* $p \in X$ *the map*

$$df_p : T_pX \to T_qY \,, \quad q = f(p)$$

is injective. Then there exists an open neighborhood, U*, of* p *in* X*, an open neighborhood,* V*, of* $f(U)$ *in* Y *and parametrizations*

$$\varphi_0 : (U_0, 0) \to (U, p)$$

and

$$\psi_0 : (V_0, 0) \to (V, q)$$

such that in the diagram below

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ {\scriptstyle \varphi_0}\uparrow & & \uparrow{\scriptstyle \psi_0} \\ U_0 & \longrightarrow & V_0 \end{array}$$

the bottom arrow, $\psi_0 \circ f \circ \varphi_0$, is the canonical immersion, ι.

Proof. The proof is identical with the proof of Theorem 4.2.6 except for the last step. In the last step one converts g into the canonical immersion via a map $\psi_1 : (V_1, 0) \to (V_0, 0)$ with the property $g \circ \psi_1 = \iota$ and then replaces ψ_0 by $\psi_0 \circ \psi_1$.

□

Exercises.

1. What is the tangent space to the quadric, $x_n = x_1^2 + \cdots + x_{n-1}^2$, at the point, $(1, 0, \ldots, 0, 1)$?

2. Show that the tangent space to the $(n-1)$-sphere, S^{n-1}, at p, is the space of vectors, $(p, \mathrm{v}) \in T_p\mathbb{R}^n$ satisfying $p \cdot \mathrm{v} = 0$.

3. Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a $\mathcal{C}^\infty$ map and let $X = \mathrm{graph} f$. What is the tangent space to X at $(a, f(a))$?

4. Let $\sigma : S^{n-1} \to S^{n-1}$ be the antipodal map, $\sigma(x) = -x$. What is the derivative of σ at $p \in S^{n-1}$?

5. Let $X_i \subseteq \mathbb{R}^{N_i}$, $i = 1, 2$, be an n_i-dimensional manifold and let $p_i \in X_i$. Define X to be the Cartesian product

$$X_1 \times X_2 \subseteq \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$$

and let $p = (p_1, p_2)$. Show that T_pX is the vector space sum of the vector spaces,$T_{p_1}X_1$ and $T_{p_2}X_2$.

6. Let $X \subseteq \mathbb{R}^N$ be an n-dimensional manifold and $\varphi_i : U_i \to X \cap V_i$, $i = 1, 2$, two parametrizations. From these parametrizations one gets an overlap diagram

$$\begin{tikzcd} & X \cap V & \\ W_1 \arrow[ur, "d_1"] \arrow[rr, "\psi"] & & W_2 \arrow[ul, "d_2"'] \end{tikzcd} \tag{4.2.11}$$

where $V = V_1 \cap V_2$, $W_i = \varphi_i^{-1}(X \cap V)$ and $\psi = \varphi_2^{-1} \circ \varphi_1$.

(a) Let $p \in X \cap V$ and let $q_i = \varphi_i^{-1}(p)$. Derive from the overlap diagram (4.2.11) an overlap diagram of linear maps

$$\begin{tikzcd} & T_p\mathbb{R}^N \arrow[dr, "(d\varphi_2)_{q_2}"] & \\ T_{q_1}\mathbb{R}^n \arrow[ur, "(d\varphi_1)_{q_1}"] \arrow[rr, "(d\psi)_{q_1}"] & & T_{q_2}\mathbb{R}^n \end{tikzcd} \tag{4.2.12}$$

(b) Use overlap diagrams to give another proof that T_pX is intrinsically defined.

4.3 Vector fields and differential forms on manifolds

A vector field on an open subset, U, of $\mathbb{R}^n$ is a function, v, which assigns to each $p \in U$ an element, $v(p)$, of T_pU, and a k-form is a function, ω, which assigns to each $p \in U$ an element, $\omega(p)$, of $\Lambda^k(T_p^*)$. These definitions have obvious generalizations to manifolds:

Definition 4.3.1. *Let X be a manifold. A vector field on X is a function, v, which assigns to each $p \in X$ an element, $v(p)$, of T_pX, and a k-form is a function, ω, which assigns to each $p \in X$ an element, $\omega(p)$, of $\Lambda^k(T_p^*X)$.*

We'll begin our study of vector fields and k-forms on manifolds by showing that, like their counterparts on open subsets of $\mathbb{R}^n$, they have nice pull-back and push-forward properties with respect to mappings. Let X and Y be manifolds and $f : X \to Y$ a $\mathcal{C}^\infty$ mapping.

Definition 4.3.2. *Given a vector field, v, on X and a vector field, w, on Y, we'll say that v and w are f-related if, for all $p \in X$ and $q = f(p)$*

$$(df)_p v(p) = w(q)\,. \tag{4.3.1}$$

In particular, if f is a diffeomorphism, and we're given a vector field, v, on X we can define a vector field, w, on Y by requiring that for every point, $q \in Y$, the identity (4.2.1) holds at the point, $p = f^{-1}(q)$. In this case we'll call w the *push-forward* of v by f and denote it by $f_* v$. Similarly, given a vector field, w, on Y we can define a vector field, v, on X by applying the same construction to the inverse diffeomorphism, $f^{-1} : Y \to X$. We will call the vector field $(f^{-1})_* w$ the *pull-back* of w by f (and also denote it by $f^* w$).

For differential forms the situation is even nicer. Just as in §2.5 we can define the pull-back operation on forms for *any* $\mathcal{C}^\infty$ map $f : X \to Y$. Specifically: Let ω be a k-form on Y. For every $p \in X$, and $q = f(p)$ the linear map

$$df_p : T_p X \to T_q Y$$

induces by (1.8.2) a pull-back map

$$(df_p)^* : \Lambda^k(T_q^*) \to \Lambda^k(T_p^*)$$

and, as in §2.5, we'll define the pull-back, $f^*\omega$, of ω to X by defining it at p by the identity

$$(f^*\omega)(p) = (df_p)^* \omega(q) . \tag{4.3.2}$$

The following results about these operations are proved in exactly the same way as in §2.5.

Proposition 4.3.3. *Let X, Y and Z be manifolds and $f : X \to Y$ and $g : Y \to Z$ $\mathcal{C}^\infty$ maps. Then if ω is a k-form on Z*

$$f^*(g^*\omega) = (g \circ f)^* \omega , \tag{4.3.3}$$

and if v is a vector field on X and f and g are diffeomorphisms

$$(g \circ f)_* v = g_*(f_* v) . \tag{4.3.4}$$

Our first application of these identities will be to define what one means by a "$\mathcal{C}^\infty$ vector field" and a "$\mathcal{C}^\infty$ k-form".

Let X be an n-dimensional manifold and U an open subset of X.

Definition 4.3.4. *The set U is a* parametrizable *open set if there exists an open set, U_0, in $\mathbb{R}^n$ and a diffeomorphism, $\varphi_0 : U_0 \to U$.*

In other words, U is parametrizable if there exists a parametrization having U as its image. (Note that X being a manifold means that every point is contained in a parametrizable open set.)

Now let $U \subseteq X$ be a parametrizable open set and $\varphi : U_0 \to U$ a parametrization of U.

Definition 4.3.5. *A k-form ω on U is $\mathcal{C}^\infty$ if $\varphi_0^*\omega$ is $\mathcal{C}^\infty$.*

This definition appears to depend on the choice of the parametrization, φ, but we claim it doesn't. To see this let $\varphi_1 : U_1 \to U$ be another parametrization of U and let

$$\psi : U_0 \to U_1$$

be the composite map, $\varphi_0^{-1} \circ \varphi_0$. Then $\varphi_0 = \varphi_1 \circ \psi$ and hence by Proposition 4.3.3

$$\varphi_0^*\omega = \psi^*\varphi_1^*\omega\,,$$

so by (2.5.11) $\varphi_0^*\omega$ is $\mathcal{C}^\infty$ if $\varphi_1^*\omega$ is $\mathcal{C}^\infty$. The same argument applied to ψ^{-1} shows that $\varphi_1^*\omega$ is $\mathcal{C}^\infty$ if $\varphi_0^*\omega$ is $\mathcal{C}^\infty$. Q.E.D

The notion of "$\mathcal{C}^\infty$" for vector fields is defined similarly:

Definition 4.3.6. *A vector field, v, on U is $\mathcal{C}^\infty$ if φ_0^*v is $\mathcal{C}^\infty$.*

By Proposition 4.3.3 $\varphi_0^*v = \psi^*\varphi_1^*v$, so, as above, this definition is independent of the choice of parametrization.

We now globalize these definitions.

Definition 4.3.7. *A k-form, ω, on X is $\mathcal{C}^\infty$ if, for every point $p \in X$, ω is $\mathcal{C}^\infty$ on a neighborhood of p. Similarly, a vector field, v, on X is $\mathcal{C}^\infty$ if, for every point, $p \in X$, v is $\mathcal{C}^\infty$ on a neighborhood of p.*

We will also use the identities (5.2.4) and (5.2.5) to prove the following two results.

Proposition 4.3.8. *Let X and Y be manifolds and $f : X \to Y$ a $\mathcal{C}^\infty$ map. Then if ω is a $\mathcal{C}^\infty$ k-form on Y, $f^*\omega$ is a $\mathcal{C}^\infty$ k-form on X.*

Proof. For $p \in X$ and $q = f(p)$ let $\varphi_0 : U_0 \to U$ and $\psi_0 : V_0 \to V$ be parametrizations with $p \in U$ and $q \in V$. Shrinking U if necessary we can assume that $f(U) \subseteq V$. Let $g : U_0 \to V_0$ be the map, $g = \psi_0^{-1} \circ f \circ \varphi_0$. Then $\psi_0 \circ g = f \circ \varphi_0$, so $g^*\psi_0^*\omega = \varphi_0^* f^*\omega$. Since ω is $\mathcal{C}^\infty$, $\psi_0^*\omega$ is $\mathcal{C}^\infty$, so by (2.5.11) $g^*\psi_0^*\omega$ is $\mathcal{C}^\infty$, and hence, $\varphi_0^* f^*\omega$ is $\mathcal{C}^\infty$. Thus by definition $f^*\omega$ is $\mathcal{C}^\infty$ on U.

$\square$

By exactly the same argument one proves:

Proposition 4.3.9. *If w is a $\mathcal{C}^\infty$ vector field on Y and f is a diffeomorphism, f^*w is a $\mathcal{C}^\infty$ vector field on X.*

Some notation:

1. We'll denote the space of $\mathcal{C}^\infty$ k-forms on X by $\Omega^k(X)$.

2. For $\omega \in \Omega^k(X)$ we'll define the support of ω to be the closure of the set

$$\{p \in X\,,\, \omega(p) \neq 0\}$$

and we'll denote by $\Omega^k_c(X)$ the space of completely supported k-forms.

3. For a vector field, v, on X we'll define the support of v to be the closure of the set

$$\{p \in X\,,\, v(p) \neq 0\}\,.$$

We will now review some of the results about vector fields and the differential forms that we proved in Chapter 2 and show that they have analogues for manifolds.

1. Integral curves

Let $I \subseteq \mathbb{R}$ be an open interval and $\gamma : I \to X$ a $\mathcal{C}^\infty$ curve. For $t_0 \in I$ we will call $\vec{u} = (t_0, 1) \in T_{t_0}\mathbb{R}$ the *unit* vector in $T_{t_0}\mathbb{R}$ and if $p = \gamma(t_0)$ we will call the vector

$$d\gamma_{t_0}(\vec{u}) \in T_pX$$

the *tangent vector* to γ at p. If v is a vector field on X we will say that γ is an *integral curve* of v if for all $t_0 \in I$

$$v(\gamma(t_0)) = d\gamma_{t_0}(\vec{u})\,.$$

Proposition 4.3.10. *Let X and Y be manifolds and $f : X \to Y$ a $\mathcal{C}^\infty$ map. If v and w are vector fields on X and Y which are f-related, then integral curves of v get mapped by f onto integral curves of w.*

Proof. If the curve, $\gamma : I \to X$ is an integral curve of v we have to show that $f \circ \gamma : I \to Y$ is an integral curve of w. If $\gamma(t) = p$ and $q = f(p)$ then by the chain rule

$$\begin{aligned} w(q) &= df_p(v(p)) = df_p(d\gamma_t(\vec{u})) \\ &= d(f \circ \gamma)_t(\vec{u}) \,. \end{aligned}$$

□

From this result it follows that the local existence, uniqueness and "smooth dependence on initial data" results about vector fields that we described in §2.1 are true for vector fields on manifolds. More explicitly, let U be a parametrizable open subset of X and $\varphi : U_0 \to U$ a parametrization. Since U_0 is an open subset of $\mathbb{R}^n$ these results are true for the vector field, $w = \varphi_0^* v$ and hence since w and v are φ_0-related they are true for v. In particular

Proposition 4.3.11 (local existence). *For every $p \in U$ there exists an integral curve, $\gamma(t)$, $-\epsilon < t < \epsilon$, of v with $\gamma(0) = p$.*

Proposition 4.3.12 (local uniqueness). *Let $\gamma_i : I_i \to U$ $i = 1, 2$ be integral curves of v and let $I = I_1 \cap I_2$. Suppose $\gamma_2(t) = \gamma_1(t)$ for some $t \in I$. Then there exists a unique integral curve, $\gamma : I \cup I_2 \to U$ with $\gamma = \gamma_1$ on I_1 and $\gamma = \gamma_2$ on I_2.*

Proposition 4.3.13 (smooth dependence on initial data). *For every $p \in U$ there exists a neighborhood, $\mathcal{O}$ of p in U, an interval $(-\epsilon, \epsilon)$ and a $\mathcal{C}^\infty$ map, $h : \mathcal{O} \times (-\epsilon, \epsilon) \to U$ such that for every $p \in \mathcal{O}$ the curve*

$$\gamma_p(t) = h(p, t) \,, \quad -\epsilon < t < \epsilon \,,$$

is an integral curve of v with $\gamma_p(0) = p$.

As in Chapter 2 we will say that v is *complete* if, for every $p \in X$ there exists an integral curve, $\gamma(t)$, $-\infty < t < \infty$, with $\gamma(0) = p$. In Chapter 2 we showed that one simple criterium for a vector field to be complete is that it be compactly supported. We will prove that the same is true for manifolds.

Theorem 4.3.14. *If X is compact or, more generally, if v is compactly supported, v is complete.*

Proof. It's not hard to prove this by the same argument that we used to prove this theorem for vector fields on $\mathbb{R}^n$, but we'll give a simpler proof that derives this directly from the $\mathbb{R}^n$ result. Suppose X is a submanifold of $\mathbb{R}^N$. Then for $p \in X$,

$$T_pX \subset T_p\mathbb{R}^N = \{(p, \mathrm{v}) , \quad \mathrm{v} \in \mathbb{R}^N\} ,$$

so $v(p)$ can be regarded as a pair, $(p, \mathrm{v}(p))$ where $\mathrm{v}(p)$ is in $\mathbb{R}^N$. Let

$$f_v : X \to \mathbb{R}^N \tag{4.3.5}$$

be the map, $f_v(p) = \mathrm{v}(p)$. It is easy to check that v is $\mathcal{C}^\infty$ if and only if f_v is $\mathcal{C}^\infty$. (See exercise 11.) Hence (see Appendix B) there exists a neighborhood, $\mathcal{O}$ of X and a map $g : \mathcal{O} \to \mathbb{R}^N$ extending f_v. Thus the vector field w on $\mathcal{O}$ defined by $w(q) = (q, g(q))$ extends the vector field v to $\mathcal{O}$. In other words if $\iota : X \hookrightarrow \mathcal{O}$ is the inclusion map, v and w are ι-related. Thus by Proposition 4.3.10 the integral curves of v are just integral curves of w that are contained in X.

Suppose now that v is compactly supported. Then there exists a function $\rho \in \mathcal{C}^\infty_o(\mathcal{O})$ which is 1 on the support of v, so, replacing w by ρw, we can assume that w is compactly supported. Thus w is complete. Let $\gamma(t)$, $-\infty < t < \infty$ be an integral curve of w. We will prove that if $\gamma(0) \in X$, then this curve is an integral curve of v. We first observe:

Lemma 4.3.15. *The set of points, $t \in \mathbb{R}$, for which $\gamma(t) \in X$ is both open and closed.*

Proof. If $p \notin \operatorname{supp} v$ then $w(p) = 0$ so if $\gamma(t) = p$, $\gamma(t)$ is the constant curve, $\gamma = p$, and there's nothing to prove. Thus we are reduced to showing that the set

$$\{t \in \mathbb{R} , \quad \gamma(t) \in \operatorname{supp} v\} \tag{4.3.6}$$

is both open and closed. Since $\operatorname{supp} v$ is compact this set is clearly closed. To show that it's open suppose $\gamma(t_0) \in \operatorname{supp} v$. By local existence there exist an interval $(-\epsilon + t_0 \,,\, \epsilon + t_0)$ and an integral curve, $\gamma_1(t)$, of v defined on this interval and taking the value $\gamma_1(t_0) = \gamma(t_0)$ at p. However since v and w are ι-related γ_1 is also an integral curve of w and so it has to coincide with γ on the interval $(-\epsilon + t_0 \,,\, \epsilon + t_0)$. In particular, for t on this interval, $\gamma(t) \in \operatorname{supp} v$, so the set (5.2.6) is open.

$\square$

To conclude the proof of Theorem 4.3.14 we note that since $\mathbb{R}$ is connected it follows that if $\gamma(t_0) \in X$ for some $t_0 \in \mathbb{R}$ then $\gamma(t) \in X$ for *all* $t \in \mathbb{R}$, and hence γ is an integral curve of v. Thus in particular *every* integral curve of v exists for all time, so v is complete.

□

Since w is complete it generates a one-parameter group of diffeomorphisms, $g_t : \mathcal{O} \to \mathcal{O}$, $-\infty < t < \infty$ having the property that the curve

$$g_t(p) = \gamma_p(t)\,, \quad -\infty < t < \infty$$

is the unique integral curve of w with initial point, $\gamma_p(0) = p$. But if $p \in X$ this curve is an integral curve of v, so the restriction

$$f_t = g_t \big| X$$

is a one-parameter group of diffeomorphisms of X with the property that for $p \in X$ the curve

$$f_t(p) = \gamma_p(t)\,, \quad -\infty < t < \infty$$

is the unique integral curve of v with initial point $\gamma_p(0) = p$.

2. The exterior differentiation operation

Let ω be a $\mathcal{C}^\infty$ k-form on X and $U \subset X$ a parametrizable open set. Given a parametrization, $\varphi_0 : U_0 \to U$ we define the exterior derivative, $d\omega$, of ω on X by the formula

$$d\omega = (\varphi_0^{-1})^* \, d\varphi_0^* \omega\,. \tag{4.3.7}$$

(Notice that since U_0 is an open subset of $\mathbb{R}^n$ and $\varphi_0^*\omega$ a k-form on U_0, the "d" on the right is well-defined.) We claim that this definition doesn't depend on the choice of parametrization. To see this let $\varphi_1 : U_1 \to U$ be another parametrization of U and let $\psi : U_0 \to U_1$ be the diffeomorphism, $\varphi_1^{-1} \circ \varphi_0$. Then $\varphi_0 = \varphi_1 \circ \psi$ and hence

$$\begin{aligned} d\varphi_0^*\omega &= d\psi^* \varphi_1^* \omega = \psi^* \, d\varphi_1^* \omega \\ &= \varphi_0^* (\varphi_1^{-1})^* \, d\varphi_1^* \omega \end{aligned}$$

hence

$$(\varphi_0^{-1})^* \, d\varphi_0^* \omega = (\varphi_1^{-1})^* \, d\varphi_1^* \omega$$

as claimed. We can therefore, define the exterior derivative, $d\omega$, globally by defining it to be equal to (4.3.7) on every parametrizable open set.

It's easy to see from the definition (4.3.7) that this exterior differentiation operation inherits from the exterior differentiation operation on open subsets of $\mathbb{R}^n$ the properties (2.3.2) and (2.3.3) and that for zero forms, i.e., $\mathcal{C}^\infty$ functions, $f : X \to \mathbb{R}$, df is the "intrinsic" df defined in Section 2.1, i.e., for $p \in X$ df_p is the derivative of f

$$df_p : T_pX \to \mathbb{R}$$

viewed as an element of $\Lambda^1(T_p^*X)$. Let's check that it also has the property (2.5.12).

Theorem 4.3.16. *Let X and Y be manifolds and $f : X \to Y$ a $\mathcal{C}^\infty$ map. Then for $\omega \in \Omega^k(Y)$*

$$f^* \, d\omega = df^*\omega \,. \tag{4.3.8}$$

Proof. For every $p \in X$ we'll check that this equality holds in a neighborhood of p. Let $q = f(p)$ and let U and V be parametrizable neighborhoods of p and q. Shrinking U if necessary we can assume $f(U) \subseteq V$. Given parametrizations

$$\varphi : U_0 \to U$$

and

$$\psi : V_0 \to V$$

we get by composition a map

$$g : U_0 \to V_0 \,, \quad g = \psi^{-1} \circ f \circ \varphi$$

with the property $\psi \circ g = f \circ \varphi$. Thus

$$\begin{aligned}
\varphi^* \, d(f^*\omega) &= d\varphi^* f^*\omega \quad \text{(by definition of } d) \\
&= d(f \circ \varphi)^*\omega \\
&= d(\psi \circ g)^*\omega \\
&= dg^*(\psi^*\omega) \\
&= g^* \, d\varphi^*\omega \quad \text{by (2.5.12)} \\
&= g^*\psi^* \, d\omega \quad \text{(by definition of } d) \\
&= \varphi^* f^* \, d\omega \,.
\end{aligned}$$

Hence $df^*\omega = f^* \, d\omega$.

□

3. The interior product and Lie derivative operation

Given a k-form, $\omega \in \Omega^k(X)$ and a $\mathcal{C}^\infty$vector field, w, we will define the interior product

$$\iota(v)\omega \in \Omega^{k-1}(X)\,, \tag{4.3.9}$$

as in §2.4, by setting

$$(\iota(v)\omega)_p = \iota(v_p)\omega_p$$

and the Lie derivative

$$L_v\omega \quad = \quad \Omega^k(X) \tag{4.3.10}$$

by setting

$$L_v\omega \quad = \quad \iota(v)\,d\omega + d\iota(v)\omega\,. \tag{4.3.11}$$

It's easily checked that these operations satisfy the identities (2.4.2)–(2.4.8) and (2.4.12)–(2.4.13) (since, just as in §2.4, these identities are deduced from the definitions (5.2.9) and (5.2.1) by purely formal manipulations). Moreover, if v is complete and

$$f_t : X \to X\,, \quad -\infty < t < \infty$$

is the one-parameter group of diffeomorphisms of X generated by v the Lie derivative operation can be defined by the alternative recipe

$$L_v\omega = \left(\frac{d}{dt} f_t^*\omega\right)(t=0) \tag{4.3.12}$$

as in (2.5.22). (Just as in §2.5 one proves this by showing that the operation (5.2.12) has the properties (2.12) and (2.13) and hence that it agrees with the operation (5.2.11) provided the two operations agree on zero-forms.)

Exercises.

1. Let $X \subseteq \mathbb{R}^3$ be the paraboloid, $x_3 = x_1^2 + x_2^2$ and let w be the vector field

$$w = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + 2x_3 \frac{\partial}{\partial x_3}\,.$$

(a) Show that w is tangent to X and hence defines by restriction a vector field, v, on X.

(b) What are the integral curves of v?

2. Let S^2 be the unit 2-sphere, $x_1^2+x_2^2+x_3^2=1$, in $\mathbb{R}^3$ and let w be the vector field

$$w = x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\,.$$

(a) Show that w is tangent to S^2, and hence by restriction defines a vector field, v, on S^2.

(b) What are the integral curves of v?

3. As in problem 2 let S^2 be the unit 2-sphere in $\mathbb{R}^3$ and let w be the vector field

$$w = \frac{\partial}{\partial x_3} - x_3\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right)$$

(a) Show that w is tangent to S^2 and hence by restriction defines a vector field, v, on S^2.

(b) What do its integral curves look like?

4. Let S^1 be the unit circle, $x_1^2+x_2^2=1$, in $\mathbb{R}^2$ and let $X = S^1\times S^1$ in $\mathbb{R}^4$ with defining equations

$$\begin{aligned} f_1 &= x_1^2+x_2^2-1=0 \\ f_2 &= x_3^2+x_4^2-1=0\,. \end{aligned}$$

(a) Show that the vector field

$$w = x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} + \lambda\left(x_4\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_4}\right)\,,$$

$\lambda\in\mathbb{R}$, is tangent to X and hence defines by restriction a vector field, v, on X.

(b) What are the integral curves of v?

(c) Show that $L_w f_i = 0$.

5. For the vector field, v, in problem 4, describe the one-parameter group of diffeomorphisms it generates.

6. Let X and v be as in problem 1 and let $f : \mathbb{R}^2 \to X$ be the map, $f(x_1, x_2) = (x_1, x_2, x_1^2 + x_2^2)$. Show that if u is the vector field,

$$u = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\,,$$

then $f_* u = v$.

7. Let X be a submanifold of X in $\mathbb{R}^N$ and let v and w be the vector fields on X and U. Denoting by ι the inclusion map of X into U, show that v and w are ι-related if and only if w is tangent to X and its restriction to X is v.

8. Let X be a submanifold of $\mathbb{R}^N$ and U an open subset of $\mathbb{R}^N$ containing X, and let v and w be the vector fields on X and U. Denoting by ι the inclusion map of X into U, show that v and w are ι-related if and only if w is tangent to X and its restriction to X is v.

9. An elementary result in number theory asserts

Theorem 4.3.17. *A number, $\lambda \in \mathbb{R}$, is irrational if and only if the set*

$$\{m + \lambda n\,, \quad m \text{ and } n \text{ intgers}\}$$

is a dense *subset of* $\mathbb{R}$.

Let v be the vector field in problem 4. Using the theorem above prove that if λ is irrational then for every integral curve, $\gamma(t)$, $-\infty < t < \infty$, of v the set of points on this curve is a dense subset of X.

10. Let X be an n-dimensional submanifold of $\mathbb{R}^N$. Prove that a vector field, v, on X is $\mathcal{C}^\infty$ if and only if the map, (5.2.5) is $\mathcal{C}^\infty$.

Hint: Let U be a parametrizable open subset of X and $\varphi : U_0 \to U$ a parametrization of U. Composing φ with the inclusion map $\iota : X \to \mathbb{R}^N$ one gets a map, $\iota \circ \varphi : U \to \mathbb{R}^N$. Show that if

$$\varphi^* v \quad = \quad \sum v_i \frac{\partial}{\partial x_j}$$

then

$$\varphi^* f_i = \sum \frac{\partial \varphi_i}{\partial x_j} v_j$$

where $f_1, \ldots, f_N$ are the coordinates of the map, f_v, and $\varphi_1, \ldots, \varphi_N$ the coordinates of $\iota \circ \varphi$.

11. Let v be a vector field on X and $\varphi : X \to \mathbb{R}$, a $\mathcal{C}^\infty$ function. Show that if the function

$$L_v \varphi = \iota(v)\, d\varphi \tag{4.3.13}$$

is zero φ is constant along integral curves of v.

12. Suppose that $\varphi : X \to \mathbb{R}$ is proper. Show that if $L_v \varphi = 0$, v is complete.

Hint: For $p \in X$ let $a = \varphi(p)$. By assumption, $\varphi^{-1}(a)$ is compact. Let $\rho \in \mathcal{C}_0^\infty(X)$ be a "bump" function which is one on $\varphi^{-1}(a)$ and let w be the vector field, ρv. By Theorem 4.3.14, w is complete and since

$$L_w \varphi = \iota(\rho v)\, d\varphi = \rho \iota(v)\, d\varphi = 0$$

φ is constant along integral curves of w. Let $\gamma(t)$, $-\infty < t < \infty$, be the integral curve of w with initial point, $\gamma(0) = p$. Show that γ is an integral curve of v.

4.4 Orientations

The last part of Chapter 5 will be devoted to the "integral calculus" of forms on manifolds. In particular we will prove manifold versions of two basic theorems of integral calculus on $\mathbb{R}^n$, Stokes theorem and the divergence theorem, and also develop a manifold version of degree theory. However, to extend the integral calculus to manifolds without getting involved in horrendously technical "orientation" issues we will confine ourselves to a special class of manifolds: orientable manifolds. The goal of this section will be to explain what this term means.

Definition 4.4.1. *Let X be an n-dimensional manifold. An* orientation *of X is a rule for assigning to each $p \in X$ an orientation of T_pX.*

Thus by definition 1.9.1 one can think of an orientation as a "labeling" rule which, for every $p \in X$, labels one of the two components of the set, $\Lambda^n(T_p^*X) - \{0\}$, by $\Lambda^n(T_p^*X)_+$, which we'll henceforth call the "plus" part of $\Lambda^n(T_p^*X)$, and the other component by $\Lambda^n(T_p^*X)_-$, which we'll henceforth call the "minus" part of $\Lambda^n(T_p^*X)$.

Definition 4.4.2. *An orientation of X is* smooth *if, for every $p \in X$, there exists a neighborhood, U, of p and a non-vanishing n-form, $\omega \in \Omega^n(U)$ with the property*

$$\omega_q = \Lambda^n(T_q^*X)_+ \tag{4.4.1}$$

for every $q \in U$.

Remark 4.4.3. *If we're given an orientation of X we can define another orientation by assigning to each $p \in X$ the opposite orientation to the orientation we already assigned, i.e., by switching the labels on $\Lambda^n(T_p^*)_+$ and $\Lambda^n(T_p^*)_-$. We will call this the* reversed *orientation of X. We will leave for you to check as an exercise that if X is connected and equipped with a smooth orientation, the only smooth orientations of X are the given orientation and its reversed orientation.*

Hint: *Given any smooth orientation of X the set of points where it agrees with the given orientation is open, and the set of points where it doesn't is also open. Therefore one of these two sets has to be empty.*

Note that if $\omega \in \Omega^n(X)$ is a non-vanishing n-form one gets from ω a smooth orientation of X by requiring that the "labeling rule" above satisfy

$$\omega_p \in \Lambda^n(T_p^*X)_+ \tag{4.4.2}$$

for every $p \in X$. If ω has this property we will call ω a *volume form*. It's clear from this definition that if ω_1 and ω_2 are volume forms on X then $\omega_2 = f_{2,1}\omega_1$ where $f_{2,1}$ is an everywhere positive $\mathcal{C}^\infty$ function.

Example 1.

Open subsets, U of $\mathbb{R}^n$. We will usually assign to U its *standard orientation*, by which we will mean the orientation defined by the n-form, $dx_1 \wedge \cdots \wedge dx_n$.

Example 2.

Let $f : \mathbb{R}^N \to \mathbb{R}^k$ be a $\mathcal{C}^\infty$ map. If zero is a regular value of f, the set $X = f^{-1}(0)$ is a submanifold of $\mathbb{R}^N$ of dimension, $n = N - k$, by Theorem 4.2.5. Moreover, for $p \in X$, T_pX is the kernel of the surjective map

$$df_p : T_p\mathbb{R}^N \to T_o\mathbb{R}^k$$

so we get from df_p a bijective linear map

$$T_p\mathbb{R}^N/T_pX \to T_o\mathbb{R}^k \,. \tag{4.4.3}$$

As explained in example 1, $T_p\mathbb{R}^N$ and $T_o\mathbb{R}^k$ have "standard" orientations, hence if we require that the map (5.3.3) be orientation preserving, this gives $T_p\mathbb{R}^N/T_pX$ an orientation and, by Theorem 1.9.4, gives T_pX an orientation. It's intuitively clear that since df_p varies smoothly with respect to p this orientation does as well; however, this fact requires a proof, and we'll supply a sketch of such a proof in the exercises.

Example 3.

A special case of example 2 is the n-sphere

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \,,\, x_1^2 + \cdots + x_{n+1}^2 = 1\} \,,$$

which acquires an orientation from its defining map, $f : \mathbb{R}^{n+1} \to \mathbb{R}$, $f(x) = x_1^2 + \cdots + x_{n+1}^2 - 1$.

Example 4.

Let X be an oriented submanifold of $\mathbb{R}^N$. For every $p \in X$, T_pX sits inside $T_p\mathbb{R}^N$ as a vector subspace, hence, via the identification, $T_p\mathbb{R}^N \leftrightarrow \mathbb{R}^N$ one can think of T_pX as a vector subspace of $\mathbb{R}^N$. In particular from the standard Euclidean inner product on $\mathbb{R}^N$ one gets, by restricting this inner product to vectors in T_pX, an inner product,

$$B_p : T_pX \times T_pX \to \mathbb{R}$$

on T_pX. Let σ_p be the volume element in $\Lambda^n(T_p^*X)$ associated with B_p (see §1.9, exercise 10) and let $\sigma = \sigma_X$ be the non-vanishing n-form on X defined by the assignment

$$p \in X \to \sigma_p \,.$$

In the exercises at the end of this section we'll sketch a proof of the following.

Theorem 4.4.4. *The form, σ_X, is $\mathcal{C}^\infty$ and hence, in particular, is a volume form. (We will call this form the* Riemannian *volume form.)*

Example 5. The Möbius strip. The Möbius strip is a surface in $\mathbb{R}^3$ which is *not* orientable. It is obtained from the rectangle

$$R = \{(x,y)\,;\, 0 \le x \le 1\,,\, -1 < y < 1\}$$

by gluing the ends together in the wrong way, i.e., by gluing $(1,y)$ to $(0,-y)$. It is easy to see that the Möbius strip can't be oriented by taking the standard orientation at $p = (1,0)$ and moving it along the line, $(t,0)$, $0 \le t \le 1$ to the point, $(0,0)$ (which is *also* the point, p, after we've glued the ends of the rectangle together).

We'll next investigate the "compatibility" question for diffeomorphisms between oriented manifolds. Let X and Y be n-dimensional manifolds and $f : X \to Y$ a diffeomorphism. Suppose both of these manifolds are equipped with orientations. We will say that f is *orientation preserving* if, for all $p \in X$ and $q = f(p)$ the linear map

$$df_p : T_pX \to T_qY$$

is orientation preserving. It's clear that if ω is a volume form on Y then f is orientation preserving if and only if $f^*\omega$ is a volume form on X, and from (1.9.5) and the chain rule one easily deduces

Theorem 4.4.5. *If Z is an oriented n-dimensional manifold and $g : Y \to Z$ a diffeomorphism, then if both f and g are orientation preserving, so is $g \circ f$.*

If $f : X \to Y$ is a diffeomorphism then the set of points, $p \in X$, at which the linear map,

$$df_p : T_pX \to T_qY\,, \quad q = f(p)\,,$$

is orientation preserving is open, and the set of points at which its orientation reversing is open as well. Hence if X is connected, df_p has to be orientation preserving at all points or orientation reversing at all points. In the latter case we'll say that f is *orientation reversing.*

If U is a parametrizable open subset of X and $\varphi : U_0 \to U$ a parametrization of U we'll say that this parametrization is an *oriented* parametrization if φ is orientation preserving with respect to the standard orientation of U_0 and the given orientation on U. Notice that if this parametrization isn't oriented we can convert it into one that is by replacing every connected component, V_0, of U_0 on which φ isn't orientation preserving by the open set

$$V_0^\sharp = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \,, \quad (x_1, \ldots, x_{n1}, -x_n) \in V_0\} \tag{4.4.4}$$

and replacing φ by the map

$$\psi(x_1, \ldots, x_n) = \varphi(x_1, \ldots, x_{n-1}, -x_n)\,. \tag{4.4.5}$$

If $\varphi_i : U_i \to U$, $i = 0, 1$, are oriented parametrizations of U and $\psi : U_0 \to U_1$ is the diffeomorphism, $\varphi_1^{-1} \circ \varphi_0$, then by the theorem above ψ is orientation preserving or in other words

$$\det \left[\frac{\partial \psi_i}{\partial x_j} \right] > 0 \tag{4.4.6}$$

at every point on U_0.

We'll conclude this section by discussing some orientation issues which will come up when we discuss Stokes theorem and the divergence theorem in §5.5. First a definition.

Definition 4.4.6. *An open subset, D, of X is a* smooth domain *if*

> *(a) its boundary is an $(n-1)$-dimensional submanifold of X and*
>
> *(b) the boundary of D coincides with the boundary of the closure of D.*

Examples.

1. The n-ball, $x_1^2 + \cdots + x_n^2 < 1$, whose boundary is the sphere, $x_1^2 + \cdots + x_n^2 = 1$.

2. The n-dimensional annulus,

$$1 < x_1^2 + \cdots + x_n^2 < 2$$

whose boundary consists of the spheres,

$$x_1^2 + \cdots + x_n^2 = 1 \text{ and } x_1^2 + \cdots + x_n^2 = 2\,.$$

3. Let S^{n-1} be the unit sphere, $x_1^2 + \cdots + x_2^2 = 1$ and let $D = \mathbb{R}^n - S^{n-1}$. Then the boundary of D is S^{n-1} but D is not a smooth domain since the boundary of its closure is empty.

4. The simplest example of a smooth domain is the half-space

$$\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n\,, \quad x_1 < 0\} \tag{4.4.7}$$

whose boundary

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n\,, \quad x_1 = 0\} \tag{4.4.8}$$

we can identify with $\mathbb{R}^{n-1}$ via the map,

$$(x_2, \dots, x_n) \in \mathbb{R}^{n-1} \to (0, x_2, \dots, x_n)\,.$$

We will show that every bounded domain looks locally like this example.

Theorem 4.4.7. *Let D be a smooth domain and p a boundary point of D. Then there exists a neighborhood, U, of p in X, an open set, U_0, in $\mathbb{R}^n$ and a diffeomorphism, $\psi : U_0 \to U$ such that ψ maps $U_0 \cap \mathbb{H}^n$ onto $U \cap D$.*

Proof. Let Z be the boundary of D. First we will prove:

Lemma 4.4.8. *For every $p \in Z$ there exists an open set, U, in X containing p and a parametrization*

$$\psi : U_0 \to U \tag{4.4.9}$$

of U with the property

$$\psi(U_0 \cap Bd\mathbb{H}^n) = U \cap Z\,. \tag{4.4.10}$$

Proof. X is locally diffeomorphic at p to an open subset of $\mathbb{R}^n$ so it suffices to prove this assertion for X equal to $\mathbb{R}^n$. However, if Z is an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ then by 4.2.7 there exists, for every $p \in Z$ a neighborhood, U, of p in $\mathbb{R}^n$ and a function, $\varphi \in \mathcal{C}^\infty(U)$ with the properties

$$x \in U \cap Z \Leftrightarrow \varphi(x) = 0 \tag{4.4.11}$$

and

$$d\varphi_p \neq 0\,. \tag{4.4.12}$$

Without loss of generality we can assume by (5.3.12) that

$$\frac{\partial \varphi}{\partial x_1}(p) \neq 0\,. \tag{4.4.13}$$

Hence if $\rho : U \to \mathbb{R}^n$ is the map

$$\rho(x_1, \ldots, x_n) = (\varphi(x), x_2, \ldots, x_n) \tag{4.4.14}$$

$(d\rho)_p$ is bijective, and hence ρ is locally a diffeomorphism at p. Shrinking U we can assume that ρ is a diffeomorphism of U onto an open set, U_0. By (5.3.11) and (4.4.14) ρ maps $U \cap Z$ onto $U_0 \cap Bd\mathbb{H}^n$ hence if we take ψ to be ρ^{-1}, it will have the property (5.3.10).

□

We will now prove Theorem 4.4.4. Without loss of generality we can assume that the open set, U_0, in Lemma 4.4.8 is an open ball with center at $q \in Bd\mathbb{H}^n$ and that the diffeomorphism, ψ maps q to p. Thus for $\psi^{-1}(U \cap D)$ there are three possibilities.

i. $\psi^{-1}(U \cap D) = (\mathbb{R}^n - Bd\mathbb{H}^n) \cap U_0$.

ii. $\psi^{-1}(U \cap D) = (\mathbb{R}^n - \overline{\mathbb{H}}^n) \cap U_0$.

or

iii. $\psi^{-1}(U \cap D) = \mathbb{H}^n \cap U_0$.

However, i. is excluded by the second hypothesis in Definition 5.3.5 and if ii. occurs we can rectify the situation by composing φ with the map, $(x_1, \ldots, x_n) \to (-x_1, x_2, \ldots, x_n)$.

□

Definition 4.4.9. *We will call an open set, U, with the properties above a D-adapted parametrizable open set.*

We will now show that if X is oriented and $D \subseteq X$ is a smooth domain then the boundary, Z, of D acquires from X a natural orientation. To see this we first observe

Lemma 4.4.10. *The diffeomorphism, $\psi : U_0 \to U$ in Theorem 4.4.7 can be chosen to be orientation preserving.*

Proof. If it is not, then by replacing ψ with the diffeomorphism, $\psi^\sharp(x_1, \dots, x_n) = \psi(x_1, \dots, x_{n-1}, -x_n)$, we get a D-adapted parametrization of U which *is* orientation preserving. (See (5.3.4)–(5.3.5).)

□

Let $V_0 = U_0 \cap \mathbb{R}^{n-1}$ be the boundary of $U_0 \cap \mathbb{H}^n$. The restriction of ψ to V_0 is a diffeomorphism of V_0 onto $U \cap Z$, and we will orient $U \cap Z$ by requiring that this map be an oriented parametrization. To show that this is an "intrinsic" definition, i.e., doesn't depend on the choice of ψ, we'll prove

Theorem 4.4.11. *If $\psi_i : U_i \to U$, $i = 0, 1$, are oriented parametrizations of U with the property*

$$\psi_i : U_i \cap \mathbb{H}^n \to U \cap D$$

the restrictions of ψ_i to $U_i \cap \mathbb{R}^{n-1}$ induce compatible orientations on $U \cap X$.

Proof. To prove this we have to prove that the map, $\varphi_1^{-1} \circ \varphi_0$, restricted to $U \cap Bd\mathbb{H}^n$ is an orientation preserving diffeomorphism of $U_0 \cap \mathbb{R}^{n-1}$ onto $U_1 \cap \mathbb{R}^{n-1}$. Thus we have to prove the following:

Proposition 4.4.12. *Let U_0 and U_1 be open subsets of $\mathbb{R}^n$ and $f : U_0 \to U_1$ an orientation preserving diffeomorphism which maps $U_0 \cap \mathbb{H}^n$ onto $U_1 \cap \mathbb{H}^n$. Then the restriction, g, of f to the boundary, $U_0 \cap \mathbb{R}^{n-1}$, of $U_0 \cap \mathbb{H}^n$ is an orientation preserving diffeomorphism, $g : U_0 \cap \mathbb{R}^{n-1} \to U_1 \cap \mathbb{R}^{n-1}$.*

Let $f(x) = (f_1(x), \dots, f_n(x))$. By assumption $f_1(x_1, \dots, x_n)$ is less than zero if x_1 is less than zero and equal to zero if x_1 is equal to zero, hence

$$\frac{\partial f_1}{\partial x_1}(0, x_2, \dots, x_n) \geq 0 \tag{4.4.15}$$

and

(4.4.16) $$\frac{\partial f_1}{\partial x_i}(0, x_2, \ldots, x_n) = 0\,, \quad i > 1$$

Moreover, since g is the restriction of f to the set $x_1 = 0$

(4.4.17) $$\frac{\partial f_i}{\partial x_j}(0, x_2, \ldots, x_n) = \frac{\partial g_i}{\partial x_j}(x_2, \ldots, x_1)$$

for $i, j \geq 2$. Thus on the set, $x_1 = 0$

(4.4.18) $$\det\left[\frac{\partial f_i}{\partial x_j}\right] = \frac{\partial f_1}{\partial x_1} \det\left[\frac{\partial g_i}{\partial x_j}\right]\,.$$

Since f is orientation preserving the left hand side of (4.4.18) is positive at all points $(0, x_2, \ldots, x_n) \in U_0 \cap \mathbb{R}^{n-1}$ hence by (4.4.15) the same is true for $\frac{\partial f_1}{\partial x_1}$ and $\det\left[\frac{\partial g_i}{\partial x_j}\right]$. Thus g is orientation preserving.

Remark 4.4.13. *For an alternative proof of this result see exercise 8 in §3.2 and exercises 4 and 5 in §3.6.*

We will now orient the boundary of D by requiring that for every D-adapted parametrizable open set, U, the orientation of Z coincides with the orientation of $U \cap Z$ that we described above. We will conclude this discussion of orientations by proving a global version of Proposition 5.3.11.

Proposition 4.4.14. *Let X_i, $i = 1, 2$, be an oriented manifold, $D_i \subseteq X_i$ a smooth domain and Z_i its boundary. Then if f is an orientation preserving diffeomorphism of (X_1, D_1) onto (X_2, D_2) the restriction, g, of f to Z_1 is an orientation preserving diffeomorphism of Z_1 onto Z_2.*

Let U be an open subset of X_1 and $\varphi : U_0 \to U$ an oriented D_1-compatible parametrization of U. Then if $V = f(U)$ the map $f \circ \varphi : U \to V$ is an oriented D_2-compatible parametrization of V and hence $g : U \cap Z_1 \to V \cap Z_2$ is orientation preserving.

□

Exercises.

1. Let V be an oriented n-dimensional vector space, B an inner product on V and $e_i \in V$, $i = 1, \dots, n$ an oriented orthonormal basis. Given vectors, $v_i \in V$, $i = 1, \dots, n$ show that if

$$b_{i,j} \quad = \quad B(v_i, v_j) \tag{4.4.19}$$

and

$$v_i \quad = \quad \sum a_{j,i} e_j \,, \tag{4.4.20}$$

the matrices $\mathcal{A} = [a_{i,j}]$ and $\mathcal{B} = [b_{i,j}]$ satisfy the identity:

$$\mathcal{B} = \mathcal{A}^t \mathcal{A} \tag{4.4.21}$$

and conclude that $\det \mathcal{B} = (\det \mathcal{A})^2$. (In particular conclude that $\det \mathcal{B} > 0$.)

2. Let V and W be oriented n-dimensional vector spaces. Suppose that each of these spaces is equipped with an inner product, and let $e_i \in V$, $i = 1, \dots, n$ and $f_i \in W$, $i = 1, \dots, n$ be oriented orthonormal bases. Show that if $A : W \to V$ is an orientation preserving linear mapping and $Af_i = v_i$ then

$$A^* \operatorname{vol}_V = (\det[b_{i,j}])^{\frac{1}{2}} \operatorname{vol}_W \tag{4.4.22}$$

where $\operatorname{vol}_V = e_1^* \wedge \cdots \wedge e_n^*$, $\operatorname{vol}_W = f_1^* \wedge \cdots \wedge f_n^*$ and $[b_{i,j}]$ is the matrix (4.4.19).

3. Let X be an oriented n-dimensional submanifold of $\mathbb{R}^n$, U an open subset of X, U_0 an open subset of $\mathbb{R}^n$ and $\varphi : U_0 \to U$ an oriented parametrization. Let φ_i, $i = 1, \dots, N$, be the coordinates of the map

$$U_0 \to U \hookrightarrow \mathbb{R}^N \,.$$

the second map being the inclusion map. Show that if σ is the Riemannian volume form on X then

$$\varphi^* \sigma \quad = \quad (\det[\varphi_{i,j}])^{\frac{1}{2}} \; dx_1 \wedge \cdots \wedge dx_n \tag{4.4.23}$$

where

$$\varphi_{i,j} \quad = \quad \sum_{k=1}^{N} \frac{\partial \varphi_k}{\partial x_i} \frac{\partial \varphi_k}{\partial x_j} \qquad 1 \le i, j \le n \,. \tag{4.4.24}$$

(*Hint:* For $p \in U_0$ and $q = \varphi(p)$ apply exercise 2 with $V = T_qX$, $W = T_p\mathbb{R}^n$, $A = (d\varphi)_p$ and $v_i = (d\varphi)_p \left(\frac{\partial}{\partial x_i}\right)_p$.) Conclude that σ is a $\mathcal{C}^\infty$ infinity n-form and hence that it *is* a volume form.

4. Given a $\mathcal{C}^\infty$ function $f : \mathbb{R} \to \mathbb{R}$, its graph

$$X = \{(x, f(x))\,, \quad x \in \mathbb{R}\}$$

is a submanifold of $\mathbb{R}^2$ and

$$\varphi : \mathbb{R} \to X\,, \quad x \to (x, f(x))$$

is a diffeomorphism. Orient X by requiring that φ be orientation preserving and show that if σ is the Riemannian volume form on X then

$$\varphi^*\sigma = \left(1 + \left(\frac{df}{dx}\right)^2\right)^{\frac{1}{2}} dx\,. \tag{4.4.25}$$

Hint: Exercise 3.

5. Given a $\mathcal{C}^\infty$ function $f : \mathbb{R}^n \to \mathbb{R}$ its graph

$$X = \{(x, f(x))\,, \quad x \in \mathbb{R}^n\}$$

is a submanifold of $\mathbb{R}^{n+1}$ and

$$\varphi : \mathbb{R}^n \to X\,, \quad x \to (x, f(x)) \tag{4.4.26}$$

is a diffeomorphism. Orient X by requiring that φ is orientation preserving and show that if σ is the Riemannian volume form on X then

$$\varphi^*\sigma = \left(1 + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2\right)^{\frac{1}{2}} dx_1 \wedge \cdots \wedge dx_n\,. \tag{4.4.27}$$

Hints:

(a) Let $v = (c_1, \ldots, c_n) \in \mathbb{R}^n$. Show that if $C : \mathbb{R}^n \to \mathbb{R}$ is the linear mapping defined by the matrix $[c_i c_j]$ then $Cv = (\sum c_i^2)v$ and $Cw = 0$ if $w \cdot v = 0$.

(b) Conclude that the eigenvalues of C are $\lambda_1 = \sum c_i^2$ and $\lambda_2 = \cdots = \lambda_n = 0$.

(c) Show that the determinant of $I + C$ is $1 + \sum c_i^2$.

(d) Use (a)–(c) to compute the determinant of the matrix (4.4.24) where φ is the mapping (4.4.26).

6. Let V be an oriented N-dimensional vector space and $\ell_i \in V^*$, $i = 1, \dots, k$, k linearly independent vectors in V^*. Define

$$L : V \to \mathbb{R}^k$$

to be the map $v \to (\ell_1(v), \dots, \ell_k(v))$.

(a) Show that L is surjective and that the kernel, W, of L is of dimension $n = N - k$.

(b) Show that one gets from this mapping a bijective linear mapping

$$V/W \to \mathbb{R}^K \tag{4.4.28}$$

and hence from the standard orientation on $\mathbb{R}^k$ an induced orientation on V/W and on W. *Hint:* §1.2, exercise 8 and Theorem 1.9.4.

(c) Let ω be an element of $\Lambda^N(V^*)$. Show that there exists a $\mu \in \Lambda^n(V^*)$ with the property

$$\ell_1 \wedge \cdots \wedge \ell_k \wedge \mu = \omega\,. \tag{4.4.29}$$

Hint: Choose an oriented basis, $e_1, \dots, e_N$ of V such that $\omega = e_1^* \wedge \cdots \wedge e_N^*$ and $\ell_i = e_i^*$ for $i = 1, \dots, k$, and let $\mu = e_{i+1}^* \wedge \cdots \wedge e_N^*$.

(d) Show that if ν is an element of $\Lambda^n(V^*)$ with the property

$$\ell_1 \wedge \cdots \wedge \ell_k \wedge \nu = 0$$

then there exist elements, ν_i, of $\Lambda^{n-1}(V^*)$ such that

$$\nu = \sum \ell_i \wedge \nu_i\,. \tag{4.4.30}$$

Hint: Same hint as in part (c).

(e) Show that if $\mu = \mu_i$, $i = 1, 2$, are elements of $\Lambda^n(V^*)$ with the property (4.4.29) and $\iota : W \to V$ is the inclusion map then $\iota^*\mu_1 = \iota^*\mu_2$. *Hint:* Let $\nu = \mu_1 - \mu_2$. Conclude from part (d) that $\iota^*\nu = 0$.

(f) Conclude that if μ is an element of $\Lambda^n(V^*)$ satisfying (4.4.29) the element, $\sigma = \iota^*\mu$, of $\Lambda^n(W^*)$ is *intrinsically* defined independent of the choice of μ.

(g) Show that σ lies in $\Lambda^n(V^*)_+$.

7. Let U be an open subset of $\mathbb{R}^N$ and $f : U \to \mathbb{R}^k$ a $\mathcal{C}^\infty$ map. If zero is a regular value of f, the set, $X = f^{-1}(0)$ is a manifold of dimension $n = N - k$. Show that this manifold has a natural smooth orientation. Some suggestions:

(a) Let $f = (f_1, \ldots, f_k)$ and let

$$df_1 \wedge \cdots \wedge df_k = \sum f_I \, dx_I$$

summed over multi-indices which are strictly increasing. Show that for every $p \in X$ $f_I(p) \neq 0$ for some multi-index, $I = (i_1, \ldots, i_k)$, $1 \leq i_1 < \cdots < i_k \leq N$.

(b) Let $J = (j_1, \ldots, j_n)$, $1 \leq j_1 < \cdots < j_n \leq N$ be the complementary multi-index to I, i.e., $j_r \neq i_s$ for all r and s. Show that

$$df_1 \wedge \cdots \wedge df_k \wedge dx_J = \pm f_I \, dx_1 \wedge \cdots \wedge dx_N$$

and conclude that the n-form

$$\mu = \pm \frac{1}{f_I} \, dx_J$$

is a $\mathcal{C}^\infty$ n-form on a neighborhood of p in U and has the property:

$$df_1 \wedge \cdots \wedge df_k \wedge \mu = dx_1 \wedge \cdots \wedge dx_N \,. \tag{4.4.31}$$

(c) Let $\iota : X \to U$ be the inclusion map. Show that the assignment

$$p \in X \to (\iota^*\mu)_p$$

defines an *intrinsic* nowhere vanishing n-form

$$\sigma \in \Omega^n(X)$$

on X. *Hint:* Exercise 6.

(d) Show that the orientation of X defined by σ coincides with the orientation that we described earlier in this section. *Hint:* Same hint as above.

8. Let S^n be the n-sphere and $\iota : S^n \to \mathbb{R}^{n+1}$ the inclusion map. Show that if $\omega \in \Omega^n(\mathbb{R}^{n+1})$ is the n-form, $\omega = \sum(-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \ldots dx_{n+1}$, the n-form $\iota^*\omega \in \Omega^n(S^n)$ is the Riemannian volume form.

9. Let S^{n+1} be the $(n+1)$-sphere and let

$$S^{n+1}_+ = \{(x_1, \ldots, x_{n+2}) \in S^{n+1}, \quad x_1 < 0\}$$

be the lower hemi-sphere in S^{n+1}.

(a) Prove that S^{n+1}_+ is a smooth domain.

(b) Show that the boundary of S^{n+1}_+ is S^n.

(c) Show that the boundary orientation of S^n agrees with the orientation of S^n in exercise 8.

4.5 Integration of forms over manifolds

In this section we will show how to integrate differential forms over manifolds. In what follows X will be an oriented n-dimensional manifold and W an open subset of X, and our goal will be to make sense of the integral

$$\int_W \omega \tag{4.5.1}$$

where ω is a compactly supported n-form. We'll begin by showing how to define this integral when the support of ω is contained in a parametrizable open set, U. Let U_0 be an open subset of $\mathbb{R}^n$ and $\varphi_0 : U_0 \to U$ a parametrization. As we noted in §5.3 we can assume without loss of generality that this parametrization is oriented. Making this assumption, we'll define

$$\int_W \omega = \int_{W_0} \varphi_0^* \omega \tag{4.5.2}$$

where $W_0 = \varphi_0^{-1}(U \cap W)$. Notice that if $\varphi^*\omega = f dx_1 \wedge \cdots \wedge dx_n$, then, by assumption, f is in $\mathcal{C}_0^\infty(U_0)$. Hence since

$$\int_{W_0} \varphi_0^*\omega = \int_{W_0} f dx_1 \dots dx_n$$

and since f is a bounded continuous function and is compactly supported the Riemann integral on the right is well-defined. (See Appendix B.) Moreover, if $\varphi_1 : U_1 \to U$ is another oriented parametrization of U and $\psi : U_0 \to U_1$ is the map, $\psi = \varphi_1^{-1} \circ \varphi_0$ then $\varphi_0 = \varphi_1 \circ \psi$, so by Proposition 4.3.3

$$\varphi_0^*\omega = \psi^*\varphi_1^*\omega\,.$$

Moreover, by (5.2.5) ψ is orientation preserving. Therefore since

$$W_1 = \psi(W_0) = \varphi_1^{-1}(U \cap W)$$

Theorem 3.5.2 tells us that

$$\int_{W_1} \varphi_1^*\omega = \int_{W_0} \varphi_0^*\omega\,. \tag{4.5.3}$$

Thus the definition (5.4.2) is a legitimate definition. It doesn't depend on the parametrization that we use to define the integral on the right. From the usual additivity properties of the Riemann integral one gets analogous properties for the integral (5.4.2). Namely for $\omega_i \in \Omega_c^n(U)$, $i = 1, 2$

$$\int_W \omega_1 + \omega_2 = \int_W \omega_1 + \int_W \omega_2 \tag{4.5.4}$$

and for $\omega \in \Omega_c^n(U)$ and $c \in \mathbb{R}$

$$\int_W c\omega = c \int_W \omega\,. \tag{4.5.5}$$

We will next show how to define the integral (5.4.1) for *any* compactly supported n-form. This we will do in more or less the same way that we defined improper Riemann integrals in Appendix B: by using partitions of unity. We'll begin by deriving from the partition of unity theorem in Appendix B a manifold version of this theorem.

Theorem 4.5.1. *Let*

$$\mathbb{U} = \{U_\alpha\,,\ \alpha \in \mathcal{I}\} \tag{4.5.6}$$

be a covering of X be open subsets. Then there exists a family of functions, $\rho_i \in \mathcal{C}_0^\infty(X)$, $i = 1, 2, 3, \ldots$, with the properties

(a) $\rho_i \geq 0$.

(b) For every compact set, $C \subseteq X$ there exists a positive integer N such that if $i > N$, $\operatorname{supp} \rho_i \cap C = \emptyset$.

(c) $\sum \rho_i = 1$.

(d) For every i there exists an $\alpha \in \mathcal{I}$ such that $\operatorname{supp} \rho_i \subseteq U_\alpha$.

Remark 4.5.2. *Conditions (a)–(c) say that the ρ_i's are a partition of unity and (d) says that this partition of unity is subordinate to the covering (5.4.6).*

Proof. For each $p \in X$ and for some U_α containing a p choose an open set O_p in $\mathbb{R}^N$ with $p \in O_p$ and with

$$\overline{O_p \cap X} \subseteq U_\alpha\,. \tag{4.5.7}$$

Let $\mathcal{O}$ be the union of the O_p's, and let $\widetilde{\rho}_i \in \mathcal{C}_0^\infty(O)$, $1, 2, \ldots$, be a partition of unity subordinate to the covering of $\mathcal{O}$ by the O_p's. By (5.4.7) the restriction, ρ_i, of $\widetilde{\rho}_i$ to X has compact support and it is clear that the ρ_i's inherit from the $\widetilde{\rho}_i$'s the properties (a)–(d). ☐

Now let the covering (5.4.6) be any covering of X by parametrizable open sets and let $\rho_i \in \mathcal{C}_0^\infty(X)$, $i = 1, 2, \ldots$, be a partition of unity subordinate to this covering. Given $\omega \in \Omega_c^n(X)$ we will define the integral of ω over W by the sum

$$\sum_{i=1}^{\infty} \int_W \rho_i \omega\,. \tag{4.5.8}$$

Note that since each ρ_i is supported in some U_α the individual summands in this sum are well-defined and since the support of ω is compact all but finitely many of these summands are zero by part (b) of Theorem 5.6.1. Hence the sum itself is well-defined. Let's show that this sum doesn't depend on the choice of $\mathbb{U}$ and the ρ_i's. Let $\mathbb{U}'$ be another covering of X by parametrizable open sets and ρ'_j, $j = 1, 2, \ldots$, a partition of unity subordinate to $\mathbb{U}'$. Then

$$\sum_j \int_W \rho'_j \omega = \sum_j \int_W \sum_i \rho'_j \rho_i \omega = \sum_j \left(\sum_i \int_W \rho'_j \rho_i \omega \right) \tag{4.5.9}$$

by (5.4.4). Interchanging the orders of summation and resuming with respect to the j's this sum becomes

$$\sum_i \int_W \sum_j \rho'_j \rho_i \omega$$

or

$$\sum_i \int_W \rho_i \omega \,.$$

Hence

$$\sum_i \int_W \rho'_j \omega = \sum_i \int_W \rho_i \omega \,,$$

so the two sums are the same. □

From (5.4.8) and (5.4.4) one easily deduces

Proposition 4.5.3. *For $\omega_i \in \Omega^n_c(X)$, $i = 1, 2$*

$$\int_W \omega_1 + \omega_2 = \int_W \omega_1 + \int_W \omega_2 \tag{4.5.10}$$

and for $\omega \in \Omega^n_c(X)$ and $c \in \mathbb{R}$

$$\int_W c\omega = c \int_W \omega \,. \tag{4.5.11}$$

The definition of the integral (5.4.1) depends on the choice of an orientation of X, but it's easy to see *how* it depends on this choice. We pointed out in Section 5.3 that if X is connected, there is just one way to orient it smoothly other than by its given orientation, namely by *reversing* the orientation of T_p at each point, p, and it's clear from the definitions (5.4.2) and (5.4.8) that the effect of doing this is to change the sign of the integral, i.e., to change $\int_X \omega$ to $-\int_X \omega$.

In the definition of the integral (5.4.1) we've allowed W to be an arbitrary open subset of X but required ω to be compactly supported. This integral is also well-defined if we allow ω to be an arbitrary element of $\Omega^n(X)$ but require the closure of W in X to be compact. To see this, note that under this assumption the sum (5.4.7) is still a finite sum, so the definition of the integral still makes sense, and the double sum on the right side of (5.4.9) is still a finite sum so it's still true that the definition of the integral doesn't depend on the choice of partitions of unity. In particular if the closure of W in X is compact we will define the volume of W to be the integral,

$$\operatorname{vol}(W) = \int_W \sigma_{\text{vol}}\,, \tag{4.5.12}$$

where σ_{vol} is the Riemannian volume form and if X itself is compact we'll define its volume to be the integral

$$\operatorname{vol}(X) = \int_X \sigma_{\text{vol}}\,. \tag{4.5.13}$$

We'll next prove a manifold version of the change of variables formula (3.5.1).

Theorem 4.5.4. *Let X' and X be oriented n-dimensional manifolds and $f : X' \to X$ an orientation preserving diffeomorphism. If W is an open subset of X and $W' = f^{-1}(W)$*

$$\int_{W'} f^*\omega = \int_W \omega \tag{4.5.14}$$

for all $\omega \in \Omega^n_c(X)$.

Proof. By (5.4.8) the integrand of the integral above is a finite sum of $\mathcal{C}^\infty$ forms, each of which is supported on a parametrizable open subset, so we can assume that ω itself as this property. Let V be a parametrizable open set containing the support of ω and let $\varphi_0 : U \to V$ be an oriented parameterization of V. Since f is a diffeomorphism its inverse exists and is a diffeomorphism of X onto X_1. Let $V' = f^{-1}(V)$ and $\varphi_0' = f^{-1} \circ \varphi_0$. Then $\varphi_0' : U \to V'$ is an oriented parameterization of V'. Moreover, $f \circ \varphi_0' = \varphi_0$ so if $W_0 = \varphi_0^{-1}(W)$ we have

$$W_0 = (\varphi_0')^{-1}(f^{-1}(W)) = (\varphi_0')^{-1}(W')$$

and by the chain rule we have

$$\varphi_0^*\omega = (f \circ \varphi_0')^*\omega = (\varphi_0')^* f^*\omega$$

hence

$$\int_W \omega = \int_{W_0} \varphi_0^*\omega = \int_{W_0} (\varphi_0')^*(f^*\omega) = \int_{W'} f^*\omega\,.$$

□

Exercise.

Show that if $f : X' \to X$ is orientation reversing

(4.5.15)
$$\int_{W'} f^*\omega = -\int_W \omega\,.$$

We'll conclude this discussion of "integral calculus on manifolds" by proving a preliminary version of Stokes theorem.

Theorem 4.5.5. *If μ is in $\Omega_c^{n-1}(X)$ then*

(4.5.16)
$$\int_X d\mu = 0\,.$$

Proof. Let ρ_i, $i = 1, 2, \dots$ be a partition of unity with the property that each ρ_i is supported in a parametrizable open set $U_i = U$. Replacing μ by $\rho_i\mu$ it suffices to prove the theorem for $\mu \in \Omega_c^{n-1}(U)$. Let $\varphi : U_0 \to U$ be an oriented parametrization of U. Then

$$\int_U d\mu = \int_{U_0} \varphi^*\, d\mu = \int_{U_0} d\varphi^*\mu = 0$$

by Theorem 3.3.1.

□

Exercises.

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function and let

$$X = \{(x, x_{n+1}) \in \mathbb{R}^{n+1}\,, \quad x_{n+1} = f(x)\}$$

be the graph of f. Let's orient X by requiring that the diffeomorphism

$$\varphi : \mathbb{R}^n \to X\,, \quad x \to (x, f(x))$$

be orientation preserving. Given a bounded open set U in $\mathbb{R}^n$ compute the Riemannian volume of the image

$$X_U = \varphi(U)$$

of U in X as an integral over U. *Hint:* §4.4, exercise 5.

2. Evaluate this integral for the open subset, X_U, of the paraboloid, $x_3 = x_1^2 + x_2^2$, U being the disk $x_1^2 + x_2^2 < 2$.

3. In exercise 1 let $\iota : X \hookrightarrow \mathbb{R}^{n+1}$ be the inclusion map of X onto $\mathbb{R}^{n+1}$.

(a) If $\omega \in \Omega^n(\mathbb{R}^{n+1})$ is the n-form, $x_{n+1}\, dx_1 \wedge \cdots \wedge dx_n$, what is the integral of $\iota^*\omega$ over the set X_U? Express this integral as an integral over U.

(b) Same question for $\omega = x_{n+1}^2\, dx_1 \wedge \cdots \wedge dx_n$.

(c) Same question for $\omega = dx_1 \wedge \cdots \wedge dx_n$.

4. Let $f : \mathbb{R}^n \to (0, +\infty)$ be a positive $\mathcal{C}^\infty$ function, U a bounded open subset of $\mathbb{R}^n$, and W the open set of $\mathbb{R}^{n+1}$ defined by the inequalities

$$0 < x_{n+1} < f(x_1, \dots, x_n)$$

and the condition $(x_1, \dots, x_n) \in U$.

(a) Express the integral of the $(n+1)$-form $\omega = x_{n+1}\, dx_1 \wedge \cdots \wedge dx_{n+1}$ over W as an integral over U.

(b) Same question for $\omega = x_{n+1}^2\, dx_1 \wedge \cdots \wedge dx_{n+1}$.

(c) Same question for $\omega = dx_1 \wedge \cdots \wedge dx_n$

5. Integrate the "Riemannian area" form

$$x_1\, dx_2 \wedge dx_3 + x_2\, dx_3 \wedge dx_1 + x_3\, dx_1 \wedge dx_2$$

over the unit 2-sphere S^2. (See §4.4, exercise 8.)

Hint: An easier problem: Using polar coordinates integrate $\omega = x_3\, dx_1 \wedge dx_2$ over the hemisphere, $x_3 = \sqrt{1 - x_1^2 - x_2^2}$, $x_1^2 + x_2^2 < 1$.

6. Let α be the one-form $\sum_{i=1}^n y_i\, dx_i$ in formula (2.7.2) and let $\gamma(t)$, $0 \le t \le 1$, be a trajectory of the Hamiltonian vector field (2.7.3). What is the integral of α over $\gamma(t)$?

4.6 Stokes theorem and the divergence theorem

Let X be an oriented n-dimensional manifold and $D \subseteq X$ a smooth domain. We showed in §5.3 that if Z is the boundary of D it acquires from D a natural orientation. Hence if $\iota : Z \to X$ is the inclusion map and μ is in $\Omega_c^{n-1}(X)$, the integral

$$\int_Z \iota^*\mu$$

is well-defined. We will prove:

Theorem 4.6.1 (Stokes theorem)**.** *For* $\mu \in \Omega_c^{k-1}(X)$

$$\int_Z \iota^*\mu = \int_D d\mu \,. \tag{4.6.1}$$

Proof. Let ρ_i, $i = 1, 2, \ldots$, be a partition of unity such that for each i, the support of ρ_i is contained in a parametrizable open set, $U_i = U$, of one of the following three types:

(a) $U \subseteq \operatorname{Int} D$.

(b) $U \subseteq \operatorname{Ext} D$.

(c) There exists an open subset, U_0, of $\mathbb{R}^n$ and an oriented D-adapted parametrization

$$\varphi : U_0 \to U \,. \tag{4.6.2}$$

Replacing μ by the finite sum $\sum \rho_i\mu$ it suffices to prove (5.5.1) for each $\rho_i\mu$ separately. In other words we can assume that the support of μ itself is contained in a parametrizable open set, U, of type (a), (b) or (c). But if U is of type (a)

$$\int_D d\mu = \int_U d\mu = \int_X d\mu$$

and $\iota^*\mu = 0$. Hence the left hand side of (5.5.1) is zero and, by Theorem 5.4.5, the right hand side is as well. If U is of type (b) the situation is even simpler: $\iota^*\mu$ is zero and the restriction of μ to D is zero, so both sides of (5.5.1) are automatically zero. Thus one is reduced to proving (5.5.1) when U is an open subset of type (c).

In this case the restriction of the map (5.5.1) to $U_0 \cap Bd\mathbb{H}^n$ is an orientation preserving diffeomorphism

$$\psi : U_0 \cap Bd\mathbb{H}^n \to U \cap Z \tag{4.6.3}$$

and

$$\iota_Z \circ \psi = \varphi \circ \iota_{\mathbb{R}^{n-1}} \tag{4.6.4}$$

where the maps $\iota = \iota_Z$ and

$$\iota_{\mathbb{R}^{n-1}} : \mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n$$

are the inclusion maps of Z into X and $Bd\mathbb{H}^n$ into $\mathbb{R}^n$. (Here we're identifying $Bd\mathbb{H}^n$ with $\mathbb{R}^{n-1}$.) Thus

$$\int_D d\mu = \int_{\mathbb{H}^n} \varphi^* \, d\mu = \int_{\mathbb{H}^n} d\varphi^* \mu$$

and by (5.5.4)

$$\begin{aligned} \int_Z \iota_Z^* \mu &= \int_{\mathbb{R}^{n-1}} \psi^* \iota_Z^* \mu \\ &= \int_{\mathbb{R}^{n-1}} \iota_{\mathbb{R}^{n-1}}^* \varphi^* \mu \\ &= \int_{Bd\mathbb{H}^n} \iota_{\mathbb{R}^{n-1}}^* \varphi^* \mu \,. \end{aligned}$$

Thus it suffices to prove Stokes theorem with μ replaced by $\varphi^* \mu$, or, in other words, to prove Stokes theorem for $\mathbb{H}^n$; and this we will now do.

Stokes theorem for $\mathbb{H}^n$: Let

$$\mu = \sum (-1)^{i-1} f_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \,.$$

Then

$$d\mu = \sum \frac{\partial f_i}{\partial x_i} \, dx_1 \wedge \cdots \wedge dx_n$$

and

$$\int_{\mathbb{H}^n} d\mu = \sum_i \int_{\mathbb{H}^n} \frac{\partial f_i}{\partial x_i} \, dx_1 \cdots dx_n \,.$$

We will compute each of these summands as an iterated integral doing the integration with respect to dx_i first. For $i > 1$ the dx_i integration ranges over the interval, $-\infty < x_i < \infty$ and hence since f_i is compactly supported

$$\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x_i}\, dx_i = f_i(x_1, \ldots, x_i, \ldots, x_n)\Big|_{x_i=-\infty}^{x_i=+\infty} = 0\,.$$

On the other hand the dx_1 integration ranges over the integral, $-\infty < x_1 < 0$ and

$$\int_{-\infty}^{0} \frac{\partial f_1}{\partial x_1}\, dx_1 = f(0, x_2, \ldots, x_n)\,.$$

Thus integrating with respect to the remaining variables we get

$$\int_{\mathbb{H}^n} d\mu = \int_{\mathbb{R}^{n-1}} f(0, x_2, \ldots, x_n)\, dx_2 \ldots\, dx_n\,. \tag{4.6.5}$$

On the other hand, since $\iota^*_{\mathbb{R}^{n-1}} x_1 = 0$ and $\iota^*_{\mathbb{R}^{n-1}}\, x_i = x_i$ for $i > 1$,

$$\iota^*_{\mathbb{R}^{n-1}} \mu = f_1(0, x_2, \ldots, x_n)\, dx_2 \wedge \cdots \wedge\, dx_n$$

so

$$\int \iota^*_{\mathbb{R}^{n-1}} \mu = \int f(0, x_2, \ldots, x_n)\, dx_2 \ldots\, dx_n\,. \tag{4.6.6}$$

Hence the two sides, (5.5.5) and (5.5.6), of Stokes theorem are equal. □

One important variant of Stokes theorem is the divergence theorem: Let ω be in $\Omega^n_c(X)$ and let v be a vector field on X. Then

$$L_v \omega = \iota(v)\, d\omega + \,d\iota(v)\omega = \,d\iota(v)\omega\,,$$

hence, denoting by ι_Z the inclusion map of Z into X we get from Stokes theorem, with $\mu = \iota(v)\omega$:

Theorem 4.6.2 (The manifold version of the divergence theorem)**.**

$$\int_D L_v \omega = \int_Z \iota^*_Z(\iota(v)\omega)\,. \tag{4.6.7}$$

If D is an open domain in $\mathbb{R}^n$ this reduces to the usual divergence theorem of multi-variable calculus. Namely if $\omega = dx_1 \wedge \cdots \wedge dx_n$ and $v = \sum v_i \frac{\partial}{\partial x_i}$ then by (2.4.14)

$$L_v \, dx_1 \wedge \cdots \wedge dx_n = \operatorname{div}(v) \, dx_1 \wedge \cdots \wedge dx_n$$

where

$$\operatorname{div}(v) = \sum \frac{\partial v_i}{\partial x_i} \,. \tag{4.6.8}$$

Thus if Z is the boundary of D and ι_Z the inclusion map of Z into $\mathbb{R}^n$

$$\int_D \operatorname{div}(v) \, dx = \int_Z \iota_Z^*(\iota_v \, dx_1 \wedge \cdots \wedge dx_n) \,. \tag{4.6.9}$$

The right hand side of this identity can be interpreted as the "flux" of the vector field, v, through the boundary of D. To see this let $f : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^\infty$ defining function for D, i.e., a function with the properties

$$p \in D \Leftrightarrow f(p) < 0 \tag{4.6.10}$$

and

$$df_p \neq 0 \text{ if } p \in BdD \,. \tag{4.6.11}$$

This second condition says that zero is a regular value of f and hence that $Z = BdD$ is defined by the non-degenerate equation:

$$p \in Z \Leftrightarrow f(p) = 0 \,.$$

Let w be the vector field

$$\left(\sum \left(\frac{\partial f}{\partial x_i} \right)^2 \right)^{-1} \sum \frac{\partial f_i}{\partial x_i} \frac{\partial}{\partial x_i} \,.$$

In view of (5.5.11) this vector field is well-defined on a neighborhood, U, of Z and satisfies

$$\iota(w) \, df = 1 \,. \tag{4.6.12}$$

Now note that since $df \wedge dx_1 \wedge \cdots \wedge dx_n = 0$

$$\begin{aligned}0 &= \iota(w)(df \wedge dx_1 \wedge \cdots \wedge dx_n)\\ &= (\iota(w)\, df)\, dx_1 \wedge \cdots \wedge dx_n - df \wedge \iota(w)\, dx_1 \wedge \cdots \wedge dx_n\\ &= dx_1 \wedge \cdots \wedge dx_n - df \wedge \iota(w)\, dx_1 \wedge \cdots \wedge dx_n\,,\end{aligned}$$

hence letting ν be the $(n-1)$-form $\iota(w)\, dx_1 \wedge \cdots \wedge dx_n$ we get the identity

$$dx_1 \wedge \cdots \wedge dx_n = df \wedge \nu \tag{4.6.13}$$

and by applying the operation, $\iota(v)$, to both sides of (5.5.13) the identity

$$\iota(v)\, dx_1 \wedge \cdots \wedge dx_n = (L_v f)\nu - df \wedge \iota(v)\nu\,. \tag{4.6.14}$$

Let $\nu_Z = \iota_Z^* \nu$ be the restriction of ν to Z. Since $\iota_Z^* = 0$, $\iota_Z^*\, df = 0$ and hence by (5.5.14)

$$\iota_Z^*(\iota(v)\, dx_1 \wedge \cdots \wedge dx_n) = \iota_Z^*(L_v f)\nu_Z\,,$$

and the formula (5.5.9) now takes the form

$$\int_D \operatorname{div}(v)\, dx = \int_Z L_v f \nu_Z \tag{4.6.15}$$

where the term on the right is by definition the flux of v through Z. In calculus books this is written in a slightly different form. Letting

$$\sigma_Z = \left(\sum \left(\frac{\partial f}{\partial x_i}\right)^2\right)^{\frac{1}{2}} \nu_Z$$

and letting

$$\vec{n} = \left(\sum \left(\frac{\partial f}{\partial x_i}\right)^2\right)^{-\frac{1}{2}} \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)$$

and

$$\vec{v} = (v_1, \ldots, v_n)$$

we have

$$L_v \nu_Z = (\vec{n} \cdot \vec{v}) \sigma_Z$$

and hence

$$\int_D \operatorname{div}(v)\, dx = \int_Z (\vec{n} \cdot \vec{v}) \sigma_Z \,. \tag{4.6.16}$$

In three dimensions σ_Z is just the standard "infinitesimal element of area" on the surface Z and n_p the unit outward normal to Z at p, so this version of the divergence theorem is the version one finds in most calculus books.

As an application of Stokes theorem, we'll give a very short alternative proof of the Brouwer fixed point theorem. As we explained in §3.6 the proof of this theorem basically comes down to proving

Theorem 4.6.3. *Let B^n be the closed unit ball in $\mathbb{R}^n$ and S^{n-1} its boundary. Then the identity map*

$$\mathrm{id}_{S^{n-1}} : S^{n-1} \to S^{n-1}$$

can't be extended to a $\mathcal{C}^\infty$ map

$$f : B^n \to S^{n-1} \,.$$

Proof. Suppose that f is such a map. Then for every $n-1$-form, $\mu \in \Omega^{n-1}(S^{n-1})$,

$$\int_{B^n} df^*\mu = \int_{S^{n-1}} (\iota_{S^{n-1}})^* f^* \mu \,. \tag{4.6.17}$$

But $df^*\mu = f^*\, d\mu = 0$ since μ is an $(n-1)$-form and S^{n-1} is an $(n-1)$-dimensional manifold, and since f is the identity map on S^{n-1}, $(\iota_{S_{n-1}})^* f^* \mu = (f \circ \iota_{S^{n-1}})^* \mu = \mu$. Thus for *every* $\mu \in \Omega^{n-1}(S^{n-1})$, (4.6.17) says that the integral of μ over S^{n-1} is zero. Since there are lots of $(n-1)$-forms for which this is not true, this shows that a mapping, f, with the property above can't exist.

□

Exercises.

1. Let B^n be the open unit ball in $\mathbb{R}^n$ and S^{n-1} the unit $(n-1)$-sphere, Show that volume $(S^{n-1}) = n\,\mathrm{volume}\,(B^n)$. *Hint:* Apply

Stokes theorem to the $(n-1)$-form $\mu = \sum(-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ and note (§5.3, exercise 9) that μ is the Riemannian volume form of S^{n-1}.

2. Let $D \subseteq \mathbb{R}^n$ be a smooth domain with boundary Z. Show that there exists a neighborhood, U, of Z in $\mathbb{R}^n$ and a $\mathcal{C}^\infty$ defining function, $g : U \to \mathbb{R}$ for D with the properties

(I) $p \in U \cap D \Leftrightarrow g(p) < 0.$

and

(II) $dg_p \neq 0$ if $p \in Z$

Hint: Deduce from Theorem 4.4.4 that a local version of this result is true. Show that you can cover Z by a family

$$\mathbb{U} = \{U_\alpha \, , \, \alpha \in \mathcal{I}\}$$

of open subsets of $\mathbb{R}^n$ such that for each there exists a function, $g_\alpha : U_\alpha \to \mathbb{R}$, with properties (I) and (II). Now let ρ_i, $i = 1, 2, \ldots,$ be a partition of unity and let $g = \sum \rho_i g_{\alpha_i}$ where $\operatorname{supp} \rho_i \subseteq U_{\alpha_i}$.

3. In exercise 2 suppose Z is compact. Show that there exists a global defining function, $f : \mathbb{R}^n \to \mathbb{R}$ for D with properties (I) and (II). *Hint:* Let $\rho \in \mathcal{C}_0^\infty(U)$, $0 \leq \rho \leq 1$, be a function which is one on a neighborhood of Z, and replace g by the function

$$f = \begin{cases} \rho g + (1-\rho) \text{ on } \operatorname{ext} D \\ g \text{ on } Z \\ \rho - g(1-\rho) \text{ on } \operatorname{int} D \, . \end{cases}$$

4. Show that the form $L_v f \nu_Z$ in formula (5.5.15) doesn't depend on what choice we make of a defining function, f, for D. *Hints:*

(a) Show that if g is another defining function then, at $p \in Z$, $df_p = \lambda \, dg_p$, where λ is a positive constant.

(b) Show that if one replaces df_p by $(dg)_p$ the first term in the product, $(L_v f)(p)(\nu_Z)_p$ changes by a factor, λ, and the second term by a factor $1/\lambda$.

5. Show that the form, ν_Z, is *intrinsically defined* in the sense that if ν is *any* $(n-1)$-form satisfying (5.5.13), ν_Z is equal to $\iota_Z^* \nu$. *Hint:* §4.5, exercise 7.

6. Show that the form, σ_Z, in the formula (5.5.16) is the Riemannian volume form on Z.

7. Show that the $(n-1)$-form

$$\mu = (x_1^2 + \cdots + x_n^2)^{-n} \sum (-1)^{r-1} x_r \, dx_1 \wedge \cdots \wedge \widehat{dx_r} \cdots dx_n$$

is closed and prove directly that Stokes theorem holds for the annulus $a < x_1^2 + \cdots + x_n^2 < b$ by showing that the integral of μ over the sphere, $x_1^2 + \cdots + x_n^2 = a$, is equal to the integral over the sphere, $x_1^2 + \cdots + x_n^2 = b$.

8. Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be an everywhere positive $\mathcal{C}^\infty$ function and let U be a bounded open subset of $\mathbb{R}^{n-1}$. Verify directly that Stokes theorem is true if D is the domain

$$0 < x_n < f(x_1, \dots, x_{n-1}) \,, \quad (x_1, \dots, x_{n-1}) \in U$$

and μ an $(n-1)$-form of the form

$$\varphi(x_1, \dots, x_n) \, dx_1 \wedge \cdots \wedge dx_{n-1}$$

where φ is in $\mathcal{C}_0^\infty(\mathbb{R}^n)$.

9. Let X be an oriented n-dimensional manifold and v a vector field on X which is complete. Verify that for $\omega \in \Omega_c^n(X)$

$$\int_X L_v \omega = 0 \,,$$

(a) directly by using the divergence theorem,

(b) indirectly by showing that

$$\int_X f_t^* \omega = \int_X \omega$$

where $f_t : X \to X$, $-\infty < t < \infty$, is the one-parameter group of diffeomorphisms of X generated by v.

10. Let X be an oriented n-dimensional manifold and $D \subseteq X$ a smooth domain whose closure is compact. Show that if Z is the boundary of D and $g : Z \to Z$ a diffeomorphism, g *can't* be extended to a smooth map, $f : D \to Z$.

4.7 Degree theory on manifolds

In this section we'll show how to generalize to manifolds the results about the "degree" of a proper mapping that we discussed in Chapter 3. We'll begin by proving the manifold analogue of Theorem 3.3.1.

Theorem 4.7.1. *Let X be an oriented connected n-dimensional manifold and $\omega \in \Omega_c^n(X)$ a compactly supported n-form. Then the following are equivalent*

(a) $\int_X \omega = 0$.

(b) $\omega = d\mu$ *for some* $\mu \in \Omega_c^{n-1}(X)$.

We've already verified the assertion (b) $\Rightarrow$ (a) (see Theorem 5.4.5), so what is left to prove is the converse assertion. The proof of this is more or less identical with the proof of the "(a) $\Rightarrow$ (b)" part of Theorem 3.2.1:

Step 1. Let U be a connected parametrizable open subset of X. If $\omega \in \Omega_c^n(U)$ has property (a), then $\omega = d\mu$ for some $\mu \in \Omega_c^{n-1}(U)$.

Proof. Let $\varphi : U_0 \to U$ be an oriented parametrization of U. Then

$$\int_{U_0} \varphi^*\omega = \int_X \omega = 0$$

and since U_0 is a connected open subset of $\mathbb{R}^n$, $\varphi^*\omega = d\nu$ for some $\nu \in \Omega_c^{n-1}(U_0)$ by Theorem 3.3.1. Let $\mu = (\varphi^{-1})^*\nu$. Then $d\mu = (\varphi^{-1})^*\, d\nu = \omega$.

□

Step 2. Fix a base point, $p_0 \in X$ and let p be any point of X. Then there exists a collection of connected parametrizable open sets, W_i, $i = 1, \ldots, N$ with $p_0 \in W_1$ and $p \in W_N$ such that, for $1 \le i \le N-1$, the intersection of W_i and W_{i+1} is non-empty.

Proof. The set of points, $p \in X$, for which this assertion is true is open and the set for which it is not true is open. Moreover, this assertion is true for $p = p_0$.

□

Step 3. We deduce Theorem 4.7.1 from a slightly stronger result. Introduce an equivalence relation on $\Omega_c^n(X)$ by declaring that two n-forms, ω_1 and ω_2, in $\Omega_c^n(X)$ are *equivalent* if $\omega_1 - \omega_2 \in d\Omega_x^{n-1}(X)$. Denote this equivalence relation by a wiggly arrow: $\omega_1 \sim \omega_2$. We will prove

Theorem 4.7.2. *For ω_1 and $\omega_2 \in \Omega_c^n(X)$ the following are equivalent*

(a) $\int_X \omega_1 = \int_X \omega_2$

(b) $\omega_1 \sim \omega_2$.

Applying this result to a form, $\omega \in \Omega_c^n(X)$, whose integral is zero, we conclude that $\omega \sim 0$, which means that $\omega = d\mu$ for some $\mu \in \Omega_c^{n-1}(X)$. Hence Theorem 4.7.2 implies Theorem 4.7.1. Conversely, if $\int_X \omega_1 = \int_X \omega_2$. Then $\int_X(\omega_1 - \omega_2) = 0$, so $\omega_1 - \omega_2 = d\mu$ for some $\mu \in \Omega_c^n(X)$. Hence Theorem 4.7.1 implies Theorem 4.7.2.

Step 4. By a partition of unity argument it suffices to prove Theorem 4.7.2 for $\omega_1 \in \Omega_c^n(U_1)$ and $\omega_2 \in \Omega_c^n(U_2)$ where U_1 and U_2 are connected parametrizable open sets. Moreover, if the integrals of ω_1 and ω_2 are zero then $\omega_i = d\mu_i$ for some $\mu_i \in \Omega_c^n(U_i)$ by step 1, so in this case, the theorem is true. Suppose on the other hand that

$$\int_X \omega_1 = \int_X \omega_2 = c \neq 0\,.$$

Then dividing by c, we can assume that the integrals of ω_1 and ω_2 are both equal to 1.

Step 5. Let W_i, $i = 1, \ldots, N$ be, as in step 2, a sequence of connected parametrizable open sets with the property that the intersections, $W_1 \cap U_1$, $W_N \cap U_2$ and $W_i \cap W_{i+1}$, $i = 1, \ldots, N-1$, are all non-empty. Select n-forms, $\alpha_0 \in \Omega_c^n(U_1 \cap W_1)$, $\alpha_N \in \Omega_c^n(W_N \cap U_2)$ and $\alpha_i \in \Omega_c^n(W_i \cap W_{i+1})$, $i = 1, \ldots, N-1$ such that the integral of each α_i over X is equal to 1. By step 1 Theorem 4.7.1 is true for U_1, U_2 and the W_i's, hence Theorem 4.7.2 is true for U_1, U_2 and the W_i's, so

$$\omega_1 \sim \alpha_0 \sim \alpha_1 \sim \cdots \sim \alpha_N \sim \omega_2$$

and thus $\omega_1 \sim \omega_2$. $\square$

Just as in (3.4.1) we get as a corollary of the theorem above the following "definition–theorem" of the degree of a differentiable mapping:

Theorem 4.7.3. *Let X and Y be compact oriented n-dimensional manifolds and let Y be connected. Given a proper $\mathcal{C}^\infty$ mapping, $f : X \to Y$, there exists a topological invariant,* $\deg(f)$, *with the defining property:*

$$\int_X f^*\omega = \deg f \int_Y \omega\,. \tag{4.7.1}$$

Proof. As in the proof of Theorem 3.4.1 pick an n-form, $\omega_0 \in \Omega^n_c(Y)$, whose integral over Y is one and define the degree of f to be the integral over X of $f^*\omega_0$, i.e., set

$$\deg(f) = \int_X f^*\omega_0\,. \tag{4.7.2}$$

Now let ω be any n-form in $\Omega^n_c(Y)$ and let

$$\int_Y \omega = c\,. \tag{4.7.3}$$

Then the integral of $\omega - c\omega_0$ over Y is zero so there exists an $(n-1)$-form, μ, in $\Omega^{n-1}_c(Y)$ for which $\omega - c\omega_0 = d\mu$. Hence $f^*\omega = cf^*\omega_0 + df^*\mu$, so

$$\int_X f^*\omega = c\int_X f^*\omega_0 = \deg(f)\int_Y \omega$$

by (5.6.2) and (5.6.3).

□

It's clear from the formula (5.6.1) that the degree of f is independent of the choice of ω_0. (Just apply this formula to any $\omega \in \Omega^n_c(Y)$ having integral over Y equal to one.) It's also clear from (5.6.1) that "degree" behaves well with respect to composition of mappings:

Theorem 4.7.4. *Let Z be an oriented, connected n-dimensional manifold and $g : Y \to Z$ a proper $\mathcal{C}^\infty$ map. Then*

$$\deg g \circ f = (\deg f)(\deg g)\,. \tag{4.7.4}$$

Proof. Let ω be an element of $\Omega_c^n(Z)$ whose integral over Z is one. Then

$$\begin{aligned}\deg g\circ f = \int_X (g\circ f)^*\omega &= \int_X f^*\circ g^*\omega = \deg f \int_Y g^*\omega \\ &= (\deg f)(\deg g)\,.\end{aligned}$$

□

We will next show how to compute the degree of f by generalizing to manifolds the formula for $\deg(f)$ that we derived in §3.6.

Definition 4.7.5. *A point, $p \in X$ is a critical point of f if the map*

$$df_p : T_pX \to T_{f(p)}Y \tag{4.7.5}$$

is not bijective.

We'll denote by C_f the set of all critical points of f, and we'll call a point $q \in Y$ a *critical* value of f if it is in the image, $f(C_f)$, of C_f and a *regular* value if it's not. (Thus the set of regular values is the set, $Y - f(C_f)$.) If q is a regular value, then as we observed in §3.6, the map (5.6.5) is bijective for every $p \in f^{-1}(q)$ and hence by Theorem 4.2.5, f maps a neighborhood U_p of p diffeomorphically onto a neighborhood, V_p, of q. In particular, $U_p \cap f^{-1}(q) = p$. Since f is proper the set $f^{-1}(q)$ is compact, and since the sets, U_p, are a covering of $f^{-1}(q)$, this covering must be a finite covering. In particular the set $f^{-1}(q)$ itself has to be a finite set. As in §2.6 we can shrink the U_p's so as to insure that they have the following properties:

(i) Each U_p is a parametrizable open set.

(ii) $U_p \cap U_{p'}$ is empty for $p \neq p'$.

(iii) $f(U_p) = f(U_{p'}) = V$ for all p and p'.

(iv) V is a parametrizable open set.

(v) $f^{-1}(V) = \bigcup U_p$, $p \in f^{-1}(q)$.

To exploit these properties let ω be an n-form in $\Omega_c^n(V)$ with integral equal to 1. Then by (v):

$$\deg(f) = \int_X f^*\omega = \sum_p \int_{U_p} f^*\omega\,.$$

But $f : U_p \to V$ is a diffeomorphism, hence by (5.4.14) and (5.4.15)

$$\int_{U_p} f^*\omega = \int_V \omega$$

if $f : U_p \to V$ is orientation preserving and

$$\int_{U_p} f^*\omega = -\int_V \omega$$

if $f : U_p \to V$ is orientation reversing. Thus we've proved

Theorem 4.7.6. *The degree of f is equal to the sum*

$$\sum_{p \in f^{-1}(q)} \sigma_p \tag{4.7.6}$$

where $\sigma_p = +1$ if the map (5.6.5) is orientation preserving and $\sigma_p = -1$ if it is orientation reversing.

We will next show that Sard's Theorem is true for maps between manifolds and hence that there exist lots of regular values. We first observe that if U is a parametrizable open subset of X and V a parametrizable open neighborhood of $f(U)$ in Y, then Sard's Theorem is true for the map, $f : U \to V$ since, up to diffeomorphism, U and V are just open subsets of $\mathbb{R}^n$. Now let q be any point in Y, let B be a compact neighborhood of q, and let V be a parametrizable open set containing B. Then if $A = f^{-1}(B)$ it follows from Theorem 3.4.2 that A can be covered by a finite collection of parametrizable open sets, $U_1, \ldots, U_N$ such that $f(U_i) \subseteq V$. Hence since Sard's Theorem is true for each of the maps $f : U_i \to V$ and $f^{-1}(B)$ is contained in the union of the U_i's we conclude that *the set of regular values of f intersects the interior of B in an open dense set.* Thus, since q is an arbitrary point of Y, we've proved

Theorem 4.7.7. *If X and Y are n-dimensional manifolds and $f : X \to Y$ is a proper $\mathcal{C}^\infty$ map the set of regular values of f is an open dense subset of Y.*

Since there exist lots of regular values the formula (5.6.6) gives us an effective way of computing the degree of f. We'll next justify our assertion that $\deg(f)$ is a topological invariant of f. To do so, let's generalize to manifolds the Definition 2.5.1, of a homotopy between $\mathcal{C}^\infty$ maps.

Definition 4.7.8. *Let X and Y be manifolds and $f_i : X \to Y$, $i = 0, 1$, a $\mathcal{C}^\infty$ map. A $\mathcal{C}^\infty$ map*

$$F : X \times [0,1] \to Y \tag{4.7.7}$$

is a homotopy between f_0 and f_1 if, for all $x \in X$, $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. Moreover, if f_0 and f_1 are proper maps, the homotopy, F, is a proper *homotopy if it is proper as a $\mathcal{C}^\infty$ map, i.e., for every compact set, C, of Y, $F^{-1}(C)$ is compact.*

Let's now prove the manifold analogue of Theorem 3.6.8.

Theorem 4.7.9. *Let X and Y be oriented n-dimensional manifolds and let Y be connected. Then if $f_i : X \to Y$, $i = 0, 1,$, is a proper map and the map (5.6.4) is a property homotopy, the degrees of these maps are the same.*

Proof. Let ω be an n-form in $\Omega^n_c(Y)$ whose integral over Y is equal to 1, and let C be the support of ω. Then if F is a proper homotopy between f_0 and f_1, the set, $F^{-1}(C)$, is compact and its projection on X

$$\{x \in X\,;\,(x,t) \in F^{-1}(C) \text{ for some } t \in [0,1]\} \tag{4.7.8}$$

is compact. Let

$$f_t : X \to Y$$

be the map: $f_t(x) = F(x,t)$. By our assumptions on F, f_t is a proper $\mathcal{C}^\infty$ map. Moreover, for all t the n-form, $f_t^*\omega$ is a $\mathcal{C}^\infty$ function of t and is supported on the fixed compact set (5.6.8). Hence it's clear from the Definition 4.5.8 that the integral

$$\int_X f_t^*\omega$$

is a $\mathcal{C}^\infty$ function of t. On the other hand this integral is by definition the degree of f_t and hence by Theorem 5.6.3 is an integer, so it doesn't depend on t. In particular, $\deg(f_0) = \deg(f_1)$.

$\square$

Exercises.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be the map, $x \to x^n$. Show that $\deg(f) = 0$ if n is even and 1 if n is odd.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be the polynomial function,

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n \,,$$

where the a_i's are in $\mathbb{R}$. Show that if n is even, $\deg(f) = 0$ and if n is odd, $\deg(f) = 1$.

3. Let S^1 be the unit circle

$$\{e^{i\theta}\,, \quad 0 \le \theta < 2\pi\}$$

in the complex plane and let $f : S^1 \to S^1$ be the map, $e^{i\theta} \to e^{iN\theta}$, N being a positive integer. What's the degree of f?

4. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ and $\sigma : S^{n-1} \to S^{n-1}$ the antipodal map, $x \to -x$. What's the degree of σ?

5. Let A be an element of the group, $O(n)$ of orthogonal $n \times n$ matrices and let

$$f_A : S^{n-1} \to S^{n-1}$$

be the map, $x \to Ax$. What's the degree of f_A?

6. A manifold, Y, is *contractable* if for some point, $p_0 \in Y$, the identity map of Y onto itself is homotopic to the constant map, $f_{p_0} : Y \to Y$, $f_{p_0}(y) = p_0$. Show that if Y is an oriented contractable n-dimensional manifold and X an oriented connected n-dimensional manifold then for every proper mapping $f : X \to Y$ $\deg(f) = 0$. In particular show that if n is greater than zero and Y is compact then Y can't be contractable. *Hint:* Let f be the identity map of Y onto itself.

7. Let X and Y be oriented connected n-dimensional manifolds and $f : X \to Y$ a proper $\mathcal{C}^\infty$ map. Show that if $\deg(f) \neq 0$ f is surjective.

8. Using Sard's Theorem prove that if X and Y are manifolds of dimension k and ℓ, with $k < \ell$ and $f : X \to Y$ is a proper $\mathcal{C}^\infty$ map, then the complement of the image of X in Y is open and dense. *Hint:* Let $r = \ell - k$ and apply Sard's Theorem to the map

$$g : X \times S^r \to Y \,, \quad g(x, a) = f(x) \,.$$

9. Prove that the sphere, S^2, and the torus, $S^1 \times S^2$, are not diffeomorphic.

4.8 Applications of degree theory

The purpose of this section will be to describe a few typical applications of degree theory to problems in analysis, geometry and topology. The first of these applications will be yet another variant of the Brouwer fixed point theorem.

Application 1. Let X be an oriented $(n+1)$-dimensional manifold, $D \subseteq X$ a smooth domain and Z the boundary of D. Assume that the closure, $\bar{D} = Z \cup D$, of D is compact (and in particular that X is compact).

Theorem 4.8.1. *Let Y be an oriented connected n-dimensional manifold and $f : Z \to Y$ a $\mathcal{C}^\infty$ map. Suppose there exists a $\mathcal{C}^\infty$ map, $F : \bar{D} \to Y$ whose restriction to Z is f. Then the degree of f is zero.*

Proof. Let μ be an element of $\Omega^n_c(Y)$. Then $d\mu = 0$, so $dF^*\mu = F^*\,d\mu = 0$. On the other hand if $\iota : Z \to X$ is the inclusion map,

$$\int_D dF^*\mu = \int_Z \iota^* F^* \mu = \int_Z f^*\mu = \deg(f) \int_Y \mu$$

by Stokes theorem since $F \circ \iota = f$. Hence $\deg(f)$ has to be zero. □

Application 2. (a non-linear eigenvalue problem)

This application is a non-linear generalization of a standard theorem in linear algebra. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. If n is even, A may not have *real* eigenvalues. (For instance for the map

$$A : \mathbb{R}^2 \to \mathbb{R}^2\,, \quad (x,y) \to (-y,x)$$

the eigenvalues of A are $\pm\sqrt{-1}$.) However, if n is odd it is a standard linear algebra fact that there exists a vector, $\mathrm{v} \in \mathbb{R}^n - \{0\}$, and a $\lambda \in \mathbb{R}$ such that $A\mathrm{v} = \lambda \mathrm{v}$. Moreover replacing v by $\frac{\mathrm{v}}{|\mathrm{v}|}$ one can assume that $|\mathrm{v}| = 1$. This result turns out to be a special case of a much more general result. Let S^{n-1} be the unit $(n-1)$-sphere in $\mathbb{R}^n$ and let $f : S^{n-1} \to \mathbb{R}^n$ be a $\mathcal{C}^\infty$ map.

Theorem 4.8.2. *There exists a vector, $\mathrm{v} \in S^{n-1}$ and a number $\lambda \in \mathbb{R}$ such that $f(\mathrm{v}) = \lambda \mathrm{v}$.*

Proof. The proof will be by contradiction. If the theorem isn't true the vectors, v and $f(\mathrm{v})$, are linearly independent and hence the vector

$$g(\mathrm{v}) = f(\mathrm{v}) - (f(\mathrm{v}) \cdot \mathrm{v})\mathrm{v} \tag{4.8.1}$$

is non-zero. Let

$$h(\mathrm{v}) = \frac{g(\mathrm{v})}{|g(\mathrm{v})|}\,. \tag{4.8.2}$$

By (5.7.1)–(5.7.2), $|\mathrm{v}| = |h(\mathrm{v})| = 1$ and $\mathrm{v} \cdot h(\mathrm{v}) = 0$, i.e., v and $h(\mathrm{v})$ are both unit vectors and are perpendicular to each other. Let

$$\gamma_t : S^{n-1} \to S^{n-1}\,, \quad 0 \le t \le 1 \tag{4.8.3}$$

be the map

$$\gamma_t(\mathrm{v}) = (\cos \pi t)\mathrm{v} + (\sin \pi t)h(\mathrm{v})\,. \tag{4.8.4}$$

For $t = 0$ this map is the identity map and for $t = 1$, it is the antipodal map, $\sigma(\mathrm{v}) = \mathrm{v}$, hence (5.7.3) asserts that the identity map and the antipodal map are homotopic and therefore that the degree of the antipodal map is one. On the other hand the antipodal map is the restriction to S^{n-1} of the map, $(x_1, \dots, x_n) \to (-x_1, \dots, -x_n)$ and the volume form, ω, on S^{n-1} is the restriction to S^{n-1} of the $(n-1)$-form

$$\sum (-1)^{i-1} x_i\, dx_i \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n\,. \tag{4.8.5}$$

If we replace x_i by $-x_i$ in (5.7.5) the sign of this form changes by $(-1)^n$ hence $\sigma^*\omega = (-1)^n\omega$. Thus if n is odd, σ is an orientation reversing diffeomorphism of S^{n-1} onto S^{n-1}, so its degree is -1, and this contradicts what we just deduced from the existence of the homotopy (5.7.4).

□

From this argument we can deduce another interesting fact about the sphere, S^{n-1}, when $n-1$ is even. For $\mathrm{v} \in S^{n-1}$ the tangent space to S^{n-1} at v is just the space,

$$\{(\mathrm{v}, w)\,; \quad w \in \mathbb{R}^n\,,\ \mathrm{v} \cdot w = 0\}\,,$$

so a vector field on S^{n-1} can be viewed as a function, $g : S^{n-1} \to \mathbb{R}^n$ with the property

$$g(\mathrm{v}) \cdot \mathrm{v} = 0 \tag{4.8.6}$$

for all $\mathrm{v} \in S^{n-1}$. If this function is non-zero at all points, then, letting h be the function, (5.7.2), and arguing as above, we're led to a contradiction. Hence we conclude:

Theorem 4.8.3. *If $n-1$ is even and v is a vector field on the sphere, S^{n-1}, then there exists a point $p \in S^{n-1}$ at which $v(p) = 0$.*

Note that if $n-1$ is odd this statement is *not* true. The vector field

$$x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} + \cdots + x_{2n-1} \frac{\partial}{\partial x_{2n}} - x_{2n} \frac{\partial}{\partial x_{2n-1}} \tag{4.8.7}$$

is a counterexample. It is nowhere vanishing and at $p \in S^{n-1}$ is tangent to S^{n-1}.

Application 3. (The Jordan–Brouwer separation theorem.) Let X be a compact oriented $(n-1)$-dimensional submanifold of $\mathbb{R}^n$. In this subsection of §5.7 we'll outline a proof of the following theorem (leaving the details as a string of exercises).

Theorem 4.8.4. *If X is connected, the complement of $X : \mathbb{R}^n - X$ has exactly two connected components.*

This theorem is known as the Jordan-Brouwer separation theorem (and in two dimensions as the Jordan curve theorem). For simple, easy to visualize, submanifolds of $\mathbb{R}^n$ like the $(n-1)$-sphere this result is obvious, and for this reason it's easy to be misled into thinking of it as being a trivial (and not very interesting) result. However, for submanifolds of $\mathbb{R}^n$ like the curve in $\mathbb{R}^2$ depicted in the figure on page 214 it's much less obvious. (In ten seconds or less, is the point, p, in this figure inside this curve or outside?)

To determine whether a point, $p \in \mathbb{R}^n - X$ is inside X or outside X, one needs a topological invariant to detect the difference, and such an invariant is provided by the "winding number".

Definition 4.8.5. *For $p \in \mathbb{R}^n - X$ let*

$$\gamma_p : X \to S^{n-1} \tag{4.8.8}$$

be the map

$$\gamma_p(x) = \frac{x-p}{|x-p|}\,. \tag{4.8.9}$$

The *winding number* of X about p is the degree of this map.

Denoting this number by $W(X,p)$ we will show below that $W(X,p) = 0$ if p is outside X and $W(X,p) = \pm 1$ (depending on the orientation of X) if p is inside X, and hence that the winding number tells us which of the two components of $\mathbb{R}^n - X$, p is contained in.

Exercise 1.

Let U be a connected component of $\mathbb{R}^n - X$. Show that if p_0 and p_1 are in U, $W(X,p_0) = W(X,p_1)$.

Hints:

(a) First suppose that the line segment,

$$p_t = (1-t)p_0 + tp_1\,, \quad 0 \le t \le 1$$

lies in U. Conclude from the homotopy invariance of degree that $W(X,p_0) = W(X,p_t) = W(X,p_1)$.

(b) Show that there exists a sequence of points

$$q_i\,, \quad i = 1,\dots,N\,, \quad q_i \in U\,,$$

with $q_1 = p_0$ and $q_N = p_1$, such that the line segment joining q_i to q_{i+1} is in U.

Exercise 2.

Show that $\mathbb{R}^n - X$ has at most two connected components.

Hints:

(a) Show that if q is in X there exists a small ϵ-ball, $B_\epsilon(q)$, centered at q such that $B_\epsilon(q) - X$ has two components. (See Theorem 4.2.7.

(b) Show that if p is in $\mathbb{R}^n - X$, there exists a sequence

$$q_i\,,\ i = 1,\dots,N\,, \quad q_i \in \mathbb{R}^n - X\,,$$

such that $q_1 = p$, $q_N \in B_\epsilon(q)$ and the line segments joining q_i to q_{i+1} are in $\mathbb{R}^n - X$.

Exercise 3.

For $\mathrm{v} \in S^{n-1}$, show that $x \in X$ is in $\gamma_p^{-1}(\mathrm{v})$ if and only if x lies on the ray

$$p + t\mathrm{v}\,, \quad 0 < t < \infty\,. \tag{4.8.10}$$

Exercise 4.

Let $x \in X$ be a point on this ray. Show that

$$(d\gamma_p)_x : T_pX \to T_{\mathrm{v}}S^{n-1} \tag{4.8.11}$$

is bijective if and only if $v \notin T_pX$, i.e., if and only if the ray (4.8.10) is *not* tangent to X at x. *Hint:* $\gamma_p : X \to S^{n-1}$ is the composition of the maps

$$\tau_p : X \to \mathbb{R}^n - \{0\}\,, \quad x \to x - p\,, \tag{4.8.12}$$

and

$$\pi : \mathbb{R}^n - \{0\} \to S^{n-1}\,, \quad y \to \frac{y}{|y|}\,. \tag{4.8.13}$$

Show that if $\pi(y) = \mathrm{v}$, then the kernel of $(d\pi)_g$, is the one-dimensional subspace of $\mathbb{R}^n$ spanned by v. Conclude that if $y = x - p$ and $\mathrm{v} = y/|y|$ the composite map

$$(d\gamma_p)_x = (d\pi)_y \circ (d\tau_p)_x$$

is bijective if and only if $\mathrm{v} \notin T_xX$.

Exercise 5.

From exercises 3 and 4 conclude that v is a regular value of γ_p if and only if the ray (4.8.10) intersects X in a finite number of points and at each point of intersection is *not* tangent to X at that point.

Exercise 6.

In exercise 5 show that the map (4.8.11) is orientation preserving if the orientations of T_xX and v are compatible with the standard orientation of $T_p\mathbb{R}^n$. (See §1.9, exercise 5.)

Exercise 7.

Conclude that $\deg(\gamma_p)$ counts (with orientations) the number of points where the ray (4.8.10) intersects X.

Exercise 8.

Let $p_1 \in \mathbb{R}^n - X$ be a point on the ray (4.8.10). Show that if $\mathrm{v} \in S^{n-1}$ is a regular value of γ_p, it is a regular value of γ_{p_1} and show that the number

$$\deg(\gamma_p) - \deg(\gamma_{p_1}) = W(X,p) - W(X,p_1)$$

counts (with orientations) the number of points on the ray lying between p and p_1. *Hint:* Exercises 5 and 7.

Exercise 8.

Let $x \in X$ be a point on the ray (4.8.10). Suppose $x = p + t\mathrm{v}$. Show that if ϵ is a small positive number and

$$p_{\pm} = p + (t \pm \epsilon)\mathrm{v}$$

then

$$W(X,p_+) = W(X,p_-) \pm 1\,,$$

and from exercise 1 conclude that p_+ and p_- lie in different components of $\mathbb{R}^n - X$. In particular conclude that $\mathbb{R}^n - X$ has exactly two components.

Exercise 9.

Finally show that if p is very large the difference

$$\gamma_p(x) - \frac{p}{|p|}\,, \quad x \in X\,,$$

is very small, i.e., γ_p is *not* surjective and hence the degree of γ_p is zero. Conclude that for $p \in \mathbb{R}^n - X$, p is in the unbounded component of $\mathbb{R}^n - X$ if $W(X,p) = 0$ and in the bounded component if $W(X,p) = \pm 1$ (the "$\pm$" depending on the orientation of X).

Notice, by the way, that the proof of Jordan–Brouwer sketched above gives us an effective way of deciding whether the point, p, in Figure 4.9.2, is inside X or outside X. Draw a non-tangential ray from p. If it intersects X in an even number of points, p is outside X and if it intersects X is an odd number of points p inside.

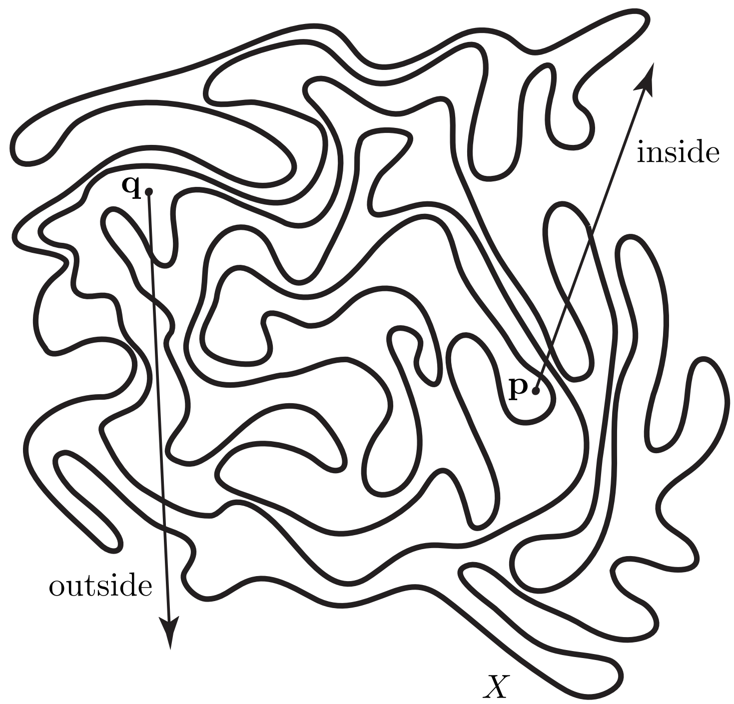


Figure 4.8.2.

Application 3. (The Gauss–Bonnet theorem.) Let $X \subseteq \mathbb{R}^n$ be a compact, connected, oriented $(n-1)$-dimensional submanifold. By the Jordan–Brouwer theorem X is the boundary of a bounded smooth domain, so for each $x \in X$ there exists a unique outward pointing unit normal vector, n_x. The Gauss map

$$\gamma : X \to S^{n-1}$$

is the map, $x \to n_x$. Let σ be the Riemannian volume form of S^{n-1}, or, in other words, the restriction to S^{n-1} of the form,

$$\sum (-1)^{i-1} x_i \, dx_1 \wedge \cdots \widehat{dx_i} \cdots \wedge \, dx_n \,,$$

and let σ_X be the Riemannian volume form of X. Then for each $p \in X$

$$(\gamma^* \sigma)_p = K(p)(\sigma_X)_q \tag{4.8.14}$$

where $K(p)$ is the *scalar curvature* of X at p. This number measures the extent to which "X is curved" at p. For instance, if X_a is the circle, $|x| = a$ in $\mathbb{R}^2$, the Gauss map is the map, $p \to p/a$, so for all p, $K_a(p) = 1/a$, reflecting the fact that, for $a < b$, X_a is more curved than X_b.

The scalar curvature can also be negative. For instance for surfaces, X in $\mathbb{R}^3$, $K(p)$ is positive at p if X is *convex* at p and negative if X is convex–concave at p. (See Figure 4.8.3 below. The surface in part (a) is convex at p, and the surface in part (b) is convex–concave.)

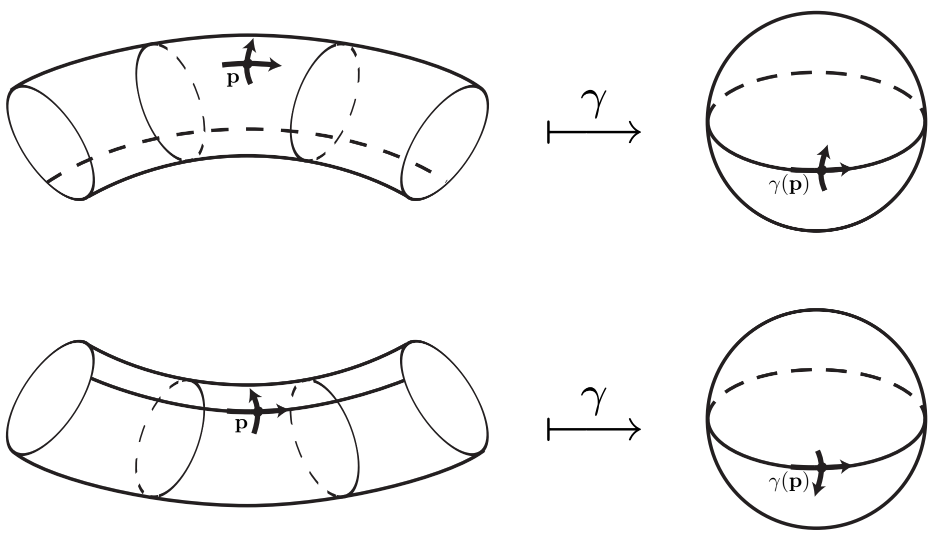


Figure 4.8.3.

Let $\mathrm{vol}\,(S^{n-1})$ be the Riemannian volume of the $(n-1)$-sphere, i.e., let

$$\mathrm{vol}\,(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

(where Γ is the gamma function). Then by (4.8.14) the quotient

$$\frac{\int K\sigma_X}{\mathrm{vol}\,(S^{n-1})} \tag{4.8.15}$$

is the degree of the Gauss map, and hence is a topological invariant of the surface of X. For $n = 3$ the *Gauss–Bonnet theorem* asserts that this topological invariant is just $1 - g$ where g is the *genus* of X or, in other words, the "number of holes". Figure 4.8.4 gives a pictorial proof of this result. (Notice that at the points, $p_1, \ldots, p_g$ the surface,X is convex–concave so the scalar curvature at these points is negative, i.e., the Gauss map is orientation reversing. On the other hand, at the point, p_0, the surface is convex, so the Gauss map at this point us orientation preserving.)

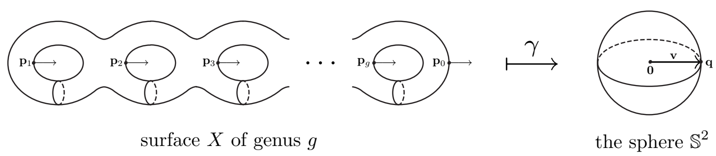


Figure 4.8.4.

4.9 Indices of Vector Fields

Let D be a bounded smooth domain in $\mathbb{R}^n$ and $v = \sum v_i \partial/\partial x_i$ a $\mathcal{C}^\infty$ vector field defined on the closure, $\bar{D}$, of D. We will define below a topological invariant which, for "generic" v's , counts (with appropriate $\pm$-signs) the number of zeroes of v. To avoid complications we'll assume v has *no* zeroes on the boundary of D.

Definition 4.9.1. *Let X be the boundary of D and let*

$$f_v : X \to S^{n-1} \tag{4.9.1}$$

be the mapping

$$p \to \frac{\mathrm{v}(p)}{|\mathrm{v}(p)|}\,,$$

where $\mathrm{v}(p) = (v_1(p), \ldots, v_n(p))$. *The* index *of* v with respect to D *is by definition the degree of this mapping.*

We'll denote this index by ind (v, D) and as a first step in investigating its properties we'll prove

Theorem 4.9.2. *Let D_1 be a smooth domain in $\mathbb{R}^n$ whose closure is contained in D. Then*

$$\mathrm{ind}(v, D) = \mathrm{ind}(v, D_1) \tag{4.9.2}$$

provided that v has no zeroes in the set $D - D_1$.

Proof. Let $W = D - \bar{D}_1$. Then the map (5.8.1) extends to a $\mathcal{C}^\infty$ map

$$F : W \to S^{n-1}, \quad p \to \frac{\mathrm{v}(p)}{|\mathrm{v}(p)|}. \tag{4.9.3}$$

Moreover,

$$Bd(W) = X \cup X_1^-$$

where X is the boundary of D with its natural boundary orientation and X_1^- is the boundary of D_1 with its boundary orientation reversed. Let ω be an element of $\Omega^{n-1}(S^{n-1})$ whose integral over S^{n-1} is 1. Then if $f = f_v$ and $f_1 = (f_1)_v$ are the restrictions of F to X and X_1 we get from Stokes theorem (and the fact that $d\omega = 0$) the identity

$$\begin{aligned} 0 &= \int_W F^* \, d\omega = \int_W dF^* \omega \\ &= \int_X f^* \omega - \int_{X_1} f_1^* \omega = \deg(f) - \deg(f_I) . \end{aligned}$$

Hence

$$\operatorname{ind}(v, D) = \deg(f) = \deg(f_1) = \deg \operatorname{ind}(v, D) .$$

□

Suppose now that v has a finite number of isolated zeroes, $p_1, \dots, p_k$, in D. Let $B_\epsilon(p_i)$ be the open ball of radius ϵ with center at p_i. By making ϵ small enough we can assume that each of these balls is contained in D and that they're mutually disjoint. We will define the *local index*, $\operatorname{ind}(v, p_i)$ of v at p_i to be the index of v with respect to $B_\epsilon(p_i)$. By Theorem 5.8.2 these local indices are unchanged if we replace ϵ by a smaller ϵ, and by applying this theorem to the domain

$$D_1 = \bigcup_{i=1}^{k} B_{p_i}(\epsilon)$$

we get, as a corollary of the theorem, the formula

$$\operatorname{ind}(v, D) = \sum_{i=1}^{k} \operatorname{ind}(v, p_i) \tag{4.9.4}$$

which computes the global index of v with respect to D in terms of these local indices.

Let's now see how to compute these local indices. Let's first suppose that the point $p = p_i$ is at the origin of $\mathbb{R}^n$. Then near $p = 0$

$$v = v_{\text{lin}} + v'$$

where

$$v = v_{\text{lin}} = \sum a_{ij} x_i \frac{\partial}{\partial xy} \tag{4.9.5}$$

the a_{ij}'s being constants and

$$v_i = \sum f_{ij} \frac{\partial}{\partial x_j} \tag{4.9.6}$$

where the f_{ij}'s vanish to second order near zero, i.e., satisfy

$$|f_{i,j}(x)| \leq C|x|^2 \tag{4.9.7}$$

for some constant, $C > 0$.

Definition 4.9.3. *We'll say that the point, $p = 0$, is a non-degenerate zero of v is the matrix, $A = [a_{i,j}]$ is non-singular.*

This means in particular that

$$\sum_j |\sum a_{i,j} x_i| \geq C_1 |x| \tag{4.9.8}$$

form some constant, $C_1 > 0$. Thus by (5.8.7) and (5.8.8) the vector field

$$v_t = v_{\text{lin}} + tv', \quad 0 \leq t \leq 1$$

has no zeroes in the ball, $B_\epsilon(p)$, $p = 0$, other than the point, $p = 0$, itself provided that ϵ is small enough. Therefore if we let X_ϵ be the boundary of $B_\epsilon(0)$ we get a homotopy

$$F : X_\epsilon \times [0,1] \to S^{n-1}, \quad (x,t) \to f v_t(x)$$

between the maps

$$f v_{\text{lin}} \quad : X_\epsilon \to S^{n-1}$$

and

$$f_v \quad : X_\epsilon \to S^{n-1}$$

and thus by Theorem 4.7.9, $\deg(f_v) = \deg(f v_{\text{lin}})$, and hence

$$\text{ind}(v,p) = \text{ind}(v_{\text{lin}\,,p})\,. \tag{4.9.9}$$

We've assumed in the above discussion that $p = 0$, but by introducing at p a translated coordinate system for which p is the origin, these comments are applicable to any zero, p of v. More explicitly if $c_1, \ldots, c_n$ are the coordinates of p then as above

$$v = v_{\text{lin}} + v'$$

where

$$v_{\text{lin}} = \sum a_{ij}(x_i - c_i)\tfrac{\partial}{\partial x_j} \tag{4.9.10}$$

and v' vanishes to second order at p, and if p is a non-degenerate zero of q, i.e., if the matrix, $A = [a_{i,j}]$, is non-singular, then exactly the same argument as before shows that

$$\text{ind}(v,p) = \text{ind}(v_{\text{lin}}\,, p)\,.$$

We will next prove:

Theorem 4.9.4. *If p is a non-degenerate zero of v, the local index, $\text{ind}(v,p)$, is $+1$ or -1 depending on whether the determinant of the matrix, $A = [a_{i,j}]$ is positive or negative.*

Proof. As above we can. without loss of generality, assume $p = 0$, and by (5.8.9) we can assume $v = v_{\text{lin}}$. Let D be the domain

$$\sum_j \Big(\sum a_{i,j}x_i\Big)^2 < 1 \tag{4.9.11}$$

and let X be its boundary. Since the only zero of v_{lin} inside this domain is $p = 0$ we get from (5.8.4) and (5.8.7)

$$\text{ind}_p(v) = \text{ind}_p(v_{\text{lin}}) = \text{ind}(v_{\text{lin}}\,, D)\,.$$

Moreover, the map

$$f_{v_{\text{lin}}} : X \to S^{n-1}$$

is, in view of (5.8.11), just the linear map, v $\to A$v, restricted to X. In particular, since A is a diffeomorphism, this mapping is as well, sothe degree of this map is $+1$ or -1 depending on whether this map is orientation preserving or not. To decide which of these alternatives is true let $\omega = \sum(-1)^i x_i\, dx_1 \wedge \cdots \widehat{dx_i} \wedge \cdots dx_n$ be the Riemannian volume form on S^{n-1} then

$$\int_X f^*_{v_{\text{lin}}}\omega = \deg(fv_{\text{lim}})\,\text{vol}\,(S^{n-1})\,. \tag{4.9.12}$$

Since v is the restriction of A to X this is equal, by Stokes theorem, to

$$\begin{aligned}
\int_D A^*\, d\omega &= n\int_D A^* dx_1 \wedge \cdots \wedge dx_n \\
&= n \det(A) \int_D dx_1 \wedge \cdots \wedge dx_n \\
&= n \det(A)\,\text{vol}\,(D)
\end{aligned}$$

which gives us the formula

$$\deg(fv_{\text{lin}}\,, D) = \frac{n \det A\,\text{vol}\,(D)}{\text{vol}\,(S^{n-1})} \tag{4.9.13}$$

□

We'll briefly describe the various types of non-degenerate zeroes that can occur for vector fields in two dimensions. To simplify this description a bit we'll assume that the matrix

$$A = \begin{bmatrix} a_{11}\,, & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is diagonalizable and that its eigenvalues are not purely imaginary. Then the following five scenarios can occur:

1. The eigenvalues of A are real and positive. In this case the integral curves of v_{lin} in a neighborhood of p look like the curves in Figure 5.8

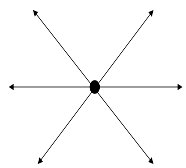


Figure 4.9.1.

and hence, is a small neighborhood of p, the integral sums of v itself look approximately like the curve in Figure 5.8.

2. The eigenvalues of A are real and negative. In this case the integral curves of v_{lin} look like the curves in Figure 5.8, but the arrows are pointing into p, rather than out of p, i.e.,

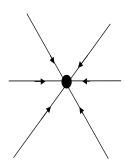


Figure 4.9.2.

3. The eigenvalues of A are real, but one is positive and one is negative. In this case the integral curves of v_{lin} look like the curves in Figure 3.

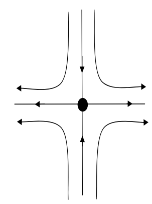


Figure 4.9.3.

4. The eigenvalues of A are complex and of the form, $a \pm \sqrt{-1}b$ with a positive. In this case the integral curves of v_{lin} are spirals going out of p as in Figure 4.

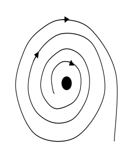


Figure 4.9.4.

5. The eigenvalues of A are complex and of the form, $a \pm \sqrt{-1}b$ with a negative. In this case the integral curves are as in Figure 4 but are spiraling into p rather than out of p.

Definition 4.9.5. *Zeroes of v of types 1 and 4 are called* sources*; those of types 2 and 5 are called* sinks *and those of type 3 are called* saddles.

Thus in particular the two-dimensional version of (5.8.4) tells us

Theorem 4.9.6. *If the vector field, v, on D has only non-degenerate zeroes then* $\mathrm{ind}(v, D)$ *is equal to the number of sources and sinks minus the number of saddles.*