Chapter 2

Differential Forms

2.1.1. Vector fields and one-forms

The goal of this chapter is to generalize to n dimensions the basic operations of three dimensional vector calculus: div, curl and grad. The "div", and "grad" operations have fairly straight-forward generalizations, but the "curl" operation is more subtle. For vector fields it doesn't have any obvious generalization, however, if one replaces vector fields by a closely related class of objects, differential forms, then not only does it have a natural generalization but it turns out that div, curl and grad are all special cases of a general operation on differential forms called *exterior differentiation*.

In this section we will discuss the simplest, easiest to understand, examples of differential forms: differential one-forms, and show that they can be regarded as dual objects to vector fields. We begin by fixing some notation.

Given $p \in \mathbb{R}^n$ we define the tangent space to $\mathbb{R}^n$ at p to be the set of pairs

$$T_p\mathbb{R}^n = \{(p, \mathrm{v})\}\,; \quad \mathrm{v} \in \mathbb{R}^n\,. \tag{2.1.1}$$

The identification

$$T_p\mathbb{R}^n \to \mathbb{R}^n\,, \quad (p, \mathrm{v}) \to \mathrm{v} \tag{2.1.2}$$

makes $T_p\mathbb{R}^n$ into a vector space. More explicitly, for v, v_1 and $\mathrm{v}_2 \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ we define the addition and scalar multiplication operations on $T_p\mathbb{R}^n$ by the recipes

$$(p, \mathrm{v}_1) + (p, \mathrm{v}_2) \quad = \quad (p, \mathrm{v}_1 + \mathrm{v}_2)$$

and

$$\lambda(p, \mathrm{v}) \quad = \quad (p, \lambda \mathrm{v})\,.$$

Let U be an open subset of $\mathbb{R}^n$ and $\varphi : U \to \mathbb{R}^m$ a C^1 map. We recall that the derivative

$$D\varphi(p) : \mathbb{R}^n \to \mathbb{R}^m$$

of φ at p is the linear map associated with the $m \times n$ matrix

$$\left[\frac{\partial \varphi_i}{\partial x_j}(p) \right] .$$

It will be useful to have a "base-pointed" version of this definition as well. Namely, if $q = \varphi(p)$ we will define

$$d\varphi_p : T_p\mathbb{R}^n \to T_q\mathbb{R}^m$$

to be the map

$$d\varphi_p(p, \mathrm{v}) = (q, D\varphi(p)\mathrm{v}) . \tag{2.1.3}$$

It's clear from the way we've defined vector space structures on $T_p\mathbb{R}^n$ and $T_q\mathbb{R}^m$ that this map is linear.

Suppose that the image of φ is contained in an open set, V, and suppose $\psi : V \to \mathbb{R}^k$ is a C^1 map. Then the "base-pointed"" version of the chain rule asserts that

$$d\psi_q \circ d\varphi_p = d(\psi \circ \varphi)_p . \tag{2.1.4}$$

(This is just an alternative way of writing $D\psi(q)D\varphi(p) = D(\psi \circ \varphi)(p)$.)

Another important vector space for us will be the vector space dual of $T_p\mathbb{R}^n$: the *cotangent space to* $\mathbb{R}^n$ *at* p. This space we'll denote by $T_p^*\mathbb{R}^n$, i.e., we'll set

$$T_p^*\mathbb{R}^n =: (T_p\mathbb{R}^n)^* .$$

Elements of this vector space come up in vector calculus as derivatives of functions. Namely if U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ a $\mathcal{C}^\infty$ function, the linear map

$$df_p : T_p\mathbb{R}^n \to T_a\mathbb{R} , \quad a = f(p) ,$$

composed with the map

$$T_a\mathbb{R} \to \mathbb{R} , \quad (a, c) \to c$$

gives one a linear map of $T_p\mathbb{R}^n$ into $\mathbb{R}$. (To avoid creating an excessive amount of fussy notation we'll continue to denote this map by df_p.) Since it is a linear map, it is by definition an element of $T_p^*\mathbb{R}^n$, and we'll call this element the *derivative* of f at p.

Example. Let $f = x_i$, the i^{th} coordinate function on $\mathbb{R}^n$. Then the derivatives

$$(dx_i)_p \qquad i = 1, \ldots, n \tag{2.1.5}$$

are a basis of the vector space, $T_p^*\mathbb{R}^n$. We will leave the verification of this as an exercise. (*Hint:* Let

$$\delta_j^i \;=\; \begin{cases} 1, & i = j \\ 0, & i = j \end{cases}$$

and let

$$e_i \;=\; (\delta_1^i, \ldots, \delta_n^i) \qquad i = 1, \ldots, n\,,$$

a "one" in the i^{th} slot and zeroes in the remaining slots. These are the standard basis vectors of $\mathbb{R}^n$, and from the formula, $(Dx_i)_p e_j = \delta_j^i$ it is easy to see that the $(dx_i)_p$'s are the basis of $T_p^*\mathbb{R}^n$ dual to the basis, (p, e_i), $i = 1, \ldots, n$ of $T_p\mathbb{R}^n$.) To have a consistent notation for these two sets of basis vectors we'll introduce the notation

$$(p, e_i) = \left(\frac{\partial}{\partial x_i}\right)_p . \tag{2.1.6}$$

(Some other notation which will be useful is the following. If U is an open subset of $\mathbb{R}^n$ and p a point of U we'll frequently write T_pU for $T_p\mathbb{R}^n$ and T_p^*U for $T_p^*\mathbb{R}^n$ when we want to emphasize that the phenomenon we're studying is taking place in U.)

We will now explain what we mean by the terms: vector field and one-form.

Definition 2.1.1. *A vector field, v, on U is a mapping which assigns to each $p \in U$ an element, $v(p)$ of T_p^*U.*

Thus $v(p)$ is a pair $(p, \mathrm{v}(p))$ where $\mathrm{v}(p)$ is an element of $\mathbb{R}^n$. From the coordinates, $\mathrm{v}_i(p)$, of $\mathrm{v}(p)$ we get functions

$$\mathrm{v}_i : U \to \mathbb{R}\,, \qquad p \to \mathrm{v}_i(p)$$

and we'll say that v is a $\mathcal{C}^\infty$ vector field if these are $\mathcal{C}^\infty$ functions. Note that by (2.1.6)

$$v(p) = \sum \mathrm{v}_i(p) \left(\frac{\partial}{\partial x_i}\right)_p . \tag{2.1.7}$$

Examples.

1. The vector field,

$$\left(\frac{\partial}{\partial x_i}\right)_p$$

This vector field we'll denote by $\frac{\partial}{\partial x_i}$.

2. Sums of vector fields: Let v_1 and v_2 be vector fields on U. Then $v_1 + v_2$ is the vector field,

$$p \in U \to v_1(p) + v_2(p) .$$

3. The multiple of a vector field by a function: Given a vector field, v on U and a function $f : U \to \mathbb{R}$ fv is the vector field

$$p \in U \to f(p)v(p) .$$

4. The vector field defined by (2.1.7): By 1–3, we can write this vector field as the sum

$$v = \sum \mathrm{v}_i \frac{\partial}{\partial x_i} . \tag{2.1.8}$$

The definition of one-form is more or less identical with the definition of vector field except that we now require our "mapping" to take its values in T_p^*U rather than T_pU.

Definition 2.1.2. *A one-form, μ, on U is a mapping which assigns to each $p \in U$ and element $\mu(p)$ of T_p^*U*

Examples.

1. Let f be a $\mathcal{C}^\infty$ function on U. Then, for each p, the derivative, df_p, is an element of T_p^*U and hence the mapping

$$df : p \in U \to df_p$$

is a one-form on U. We'll call df the *exterior* derivative of f.

2. In particular for each coordinate function, x_i, we get a one-form dx_i.

3. If μ_1 and μ_2 are one-forms on U, their sum, $\mu_1 + \mu_2$, is the one-form

$$p \in U \to (\mu_1)_p + (\mu_2)_p \,.$$

4. If f is a $\mathcal{C}^\infty$ function on U and μ a one-form, the *multiple of* μ *by* f, $f\mu$, is the one-form

$$p \in U \to f(p)\mu_p \,.$$

5. Given a one-form, μ, and a point $p \in U$ we can write μ_p as a sum

$$\mu_p = \sum f_i(p)(dx_i)_p \,, \quad f_i(p) \in \mathbb{R} \,,$$

since the $(dx_i)_p$'s are a basis of T_p^*U. Hence

$$\mu = \sum f_i \, dx_i$$

where the f_i's are the functions, $p \in U \to f_i(p)$. We'll say that μ is $\mathcal{C}^\infty$ if the f_i's are $\mathcal{C}^\infty$.

Exercise: Show that the one-form, df, in Example 1 is given by

$$df = \sum \frac{\partial x}{\partial x_i} \, dx_i \,. \tag{2.1.9}$$

Remark.

Superficially the definitions (2.1.1) and (2.1.2) look similar. However, we'll learn by the end of this chapter that the objects they define have very different properties. In fact we'll see an inkling of this difference in the definition (2.1.11) below.

Let V be an open subset if $\mathbb{R}^n$ and $\varphi : U \to V$ a $\mathcal{C}^\infty$ map. Then for $p \in U$ and $q \in \varphi(p)$ we have a map

$$d\varphi_p : T_pU \to T_qV)$$

and hence a transpose map

$$(d\varphi_p)^* : T_q^*V \to T_p^*U \,. \tag{2.1.10}$$

Thus if ν is a one-form on V, we can define a one-form, $\varphi^*\nu$, on U by setting

$$\varphi^*\nu_p = (d\varphi)_p^*\nu_q \tag{2.1.11}$$

for all $p \in U$. The form defined by this recipe is called the *pull-back* of ν by φ. In particular, if f is a $\mathcal{C}^\infty$ function on V and $\nu = df$

$$\begin{aligned}(\varphi^*\, df)_p &= (d\varphi_p)^*\, df_q \\ &= (df_q) \circ d\varphi_p = d(f \circ \varphi)_p\end{aligned}$$

by (2.1.11) and the chain rule. Hence if we define $\varphi^* f$ (the "pull-back of f by φ") to be the function, $f \circ \varphi$, we get from this computation the formula

$$\varphi^*\, df = d\varphi^* f\,. \tag{2.1.12}$$

More generally let

$$\nu = \sum_{i=1}^m f_i\, dx_i\,,$$

$f_i \in \mathcal{C}^\infty(V)$, be any $\mathcal{C}^\infty$ one-form on U. The by (2.1.11)

$$(\varphi^*\nu)_p = \sum f_i(q) d(x_i \circ \varphi)_p\,.$$

Exercise: Deduce from this that

$$\varphi^*\nu = \sum \varphi^* f_i\, d\varphi_i \tag{2.1.13}$$

where $\varphi_i = x_i \circ \varphi\,,\quad i = 1, \ldots, n$, is the i^{th} coordinate of φ.

Remark.

The pull-back operation (2.1.11) and its generalization to k-forms (see §2.6) will play a fundamental role in what follows. No analogue of this operation exists for vector fields, but in the next section we'll show that there is a weak substitution: a "push-forward operation", $\varphi_* v$, for vector fields on U. This operation, however, can only be defined if $m = n$ and φ is a diffeomorphism.

Another important operation on one-forms is the *interior product* operation. Let v be a vector field on U. Then, given a one-form, μ, on U, we can define a function

$$\iota(v)\mu : U \to \mathbb{R}$$

by setting

$$\iota(v)\mu(p) = \mu_p(v(p)) . \tag{2.1.14}$$

Notice that the right hand side makes sense since $v(p)$ is in T_pU and μ_p in its vector space dual, T_p^*U. We'll leave for you to check that if

$$\begin{aligned} v &= \sum v_i \frac{\partial}{\partial x_i} \\ \text{and} \\ \mu &= \sum f_i \, dx_i \end{aligned}$$

the function, $\iota(v)\mu$ is just the function, $\sum v_i f_i$. Hence if v and μ are $\mathcal{C}^\infty$, this function is as well. In particular, for $f \in \mathcal{C}^\infty(U)$

$$\iota(v)\, df = \sum v_i \frac{\partial f}{\partial x_i} . \tag{2.1.15}$$

Definition 2.1.3. *The expression (2.1.15) is called the Lie derivative of f by v and denoted $L_v f$.*

Exercise:

Check that for $f_i \in \mathcal{C}^\infty(U)$, $i = 1, 2$

$$L_v(f_1 f_2) = f_2 L_v f + f_1 L_v f_2 . \tag{2.1.16}$$

Exercises for §2.1

1. Verify that the co-vectors, (2.1.5) are a basis of T_p^*U.

2. Verify (2.1.9).

3. Verify (2.1.13).

4. Verify (2.1.16).

5. Let U be an open subset of $\mathbb{R}^n$ and v_1 and v_2 vector fields on U. Show that there is a unique vector field, w, on U with the property

$$L_w \varphi = L_{v_1}(L_{v_2}\varphi) - L_{v_2}(L_{v_1}\varphi)$$

for all $\varphi \in \mathcal{C}^\infty(U)$.

6. The vector field w in exercise 5 is called the *Lie bracket* of the vector fields v_1 and v_2 and is denoted $[v_1, v_2]$. Verify that "Lie bracket" satisfies the identities

$$[v_1, v_2] = -[v_2, v_1]$$

and

$$[v_1[v_2, v_3]] + [v_2[v_3, v_1]] + [v_3[v_1, v_2]] = 0\,.$$

Hint: Prove analogous identities for L_{v_1}, L_{v_2} and L_{v_3}.

7. Let $v_1 = \partial/\partial x_i$ and $v_2 = \sum g_j \partial/\partial x_j$. Show that

$$[v_1, v_2] = \sum \frac{\partial}{\partial x_i} g_i \frac{\partial}{\partial x_j}\,.$$

8. Let v_1 and v_2 be vector field and f a $\mathcal{C}^\infty$ function. Show that

$$[v_1, f v_2] = L_{v_1} f v_2 + f[v_1, v_2]\,.$$

9. Let U be an open subset of $\mathbb{R}^n$ and let $\gamma : [a, b] \to U, t \to (\gamma_1(t), \ldots, \gamma_n(t))$ be a C^1 curve. Given $\omega = \sum f_i dx_1 \in \Omega^1(U)$, define the *line integral* of ω over γ to be the integral

$$\int_\gamma \omega = \sum_{i=1}^n \int_a^b f_i(\gamma(t)) \frac{d\gamma_i}{dt}\, dt\,.$$

Show that if $\omega = df$ for some $f \in \mathcal{C}^\infty(U)$

$$\int_\gamma \omega(\gamma(b)) - f(\gamma(a))\,.$$

In particular conclude that if γ is a closed curve, i.e., $\gamma(a) = \gamma(b)$, this integral is zero.

10. Let

$$\omega = \frac{x_1 dx_2 - x_2 dx_1}{x_1^2 + x_2^2} \in \Omega^1(\mathbb{R}^2 - \{0\})\,,$$

and let $\gamma : [0, 2\pi] \to \mathbb{R}^2 - \{0\}$ be the closed curve, $t \to (\cos t \sin t)$. Compute the line integral, $\int_\gamma \omega$, and show that it's not zero. Conclude that ω can't be "d" of a function, $f \in \mathcal{C}^\infty(\mathbb{R}^2 - \{0\})$.

11. Let f be the function

$$f(x_2, x_2) = \begin{cases} \arctan \frac{x_2}{x_1} & x_1 > 0 \\ \frac{\pi}{2}, x_1 = 0 & x_2 > 0 \\ \arctan \frac{x_2}{x_1} + \pi & x_1 < 0 \end{cases}$$

where, we recall: $-\frac{\pi}{2} \arctan < \frac{\pi}{2}$. Show that this function is $\mathcal{C}^\infty$ and that df is the 1-form, ω, in the previous exercise. Why doesn't this contradict what you proved in exercise 16?

2.2 Integral curves of vector fields

In this section we'll discuss some properties of vector fields which we'll need for the manifold segment of these notes. We'll begin by generalizing to n-dimensions the calculus notion of an "integral curve" of a vector field. Let U be an open subset of $\mathbb{R}^n$ and let

$$v = \sum f_i \frac{\partial}{\partial x_i}$$

be a $\mathcal{C}^\infty$ vector field on U.

Definition 2.2.1. *A C^1 curve $\gamma : (a, b) \to U$, $\gamma(t) = (\gamma(t), \dots \gamma_n(t))$, is an integral curve of v if, for all $a < t < b$ and $p = \gamma(t)$*

$$\left(p, \frac{d\gamma}{dt}(t)\right) = v(p)$$

i.e., the condition for $\gamma(t)$ to be an integral curve of v is that it satisfy the system of differential equations

$$\frac{d\gamma_i}{dt}(t) = v_i(\gamma(t)), \quad i = 1, \dots, n. \tag{2.2.1}$$

We will quote without proof a number of basic facts about systems of ordinary differential equations of the type (2.2.1). (A source for these results that we highly recommend is Birkoff-Rota, *Ordinary Differential Equations*, Chapter 6.)

Theorem 2.2.2. *(Existence).*

Given a point $p_0 \in U$ and $b \in \mathbb{R}$, there exists an interval $I = (b - T, b + T)$, a neighborhood, U_0, of p_0 in U and for every $p \in U_0$ an integral curve, $\gamma_p : I \to U$ with $\gamma_p(b) = p$.

Theorem 2.2.3. *(Uniqueness).*

Let $\gamma_i : I_i \to U$, $i = 1, 2$, be integral curves. If $a \in I_1 \cap I_2$ and $\gamma_1(a) = \gamma_2(a)$ then $\gamma_1 \equiv \gamma_2$ on $I_1 \cap I_2$ and the curve $\gamma : I_1 \cup I_2 \to U$ defined by

$$\gamma(t) = \begin{cases} \gamma_1(t)\,, & t \in I_1 \\ \gamma_2(t)\,, & t \in I_2 \end{cases}$$

is an integral curve.

Theorem 2.2.4. *(Smooth dependence on initial data).*

Let V be an open subset of U, $I \subseteq \mathbb{R}$ an open interval, $a \in I$ a point in this interval and $h : V \times I \to U$ a mapping with the properties:

(i) $h(p, a) = p$.

(ii) For all $p \in V$ the curve

$$\gamma_p : I \to U \quad \gamma_p(t) = h(p.t)$$

is an integral curve of v.

Then the mapping, h is $\mathcal{C}^\infty$.

One important feature of the system (2.2.1) is that it is an *autonomous* system of differential equations: the function, $v_i(x)$, is a function of x alone, it doesn't depend on t. One consequence of this is the following:

Theorem 2.2.5. *Let $I = (a, b)$ and for $c \in \mathbb{R}$ let $I_c = (a - c, b - c)$. Then if $\gamma : I \to U$ is an integral curve, the reparametrized curve*

$$\gamma_c : I_c \to U\,, \quad \gamma_c(t) = \gamma(t + c) \tag{2.2.2}$$

is an integral curve.

We recall that a C^1-function $\varphi : U \to \mathbb{R}$ is an *integral* of the system (2.1.10) if for every integral curve $\gamma(t)$, the function $t \to \varphi(\gamma(t))$ is constant. This is true if and only if for all t and $p = \gamma(t)$

$$0 = \frac{d}{dt}\varphi(\gamma(t)) = (D\varphi)_p\left(\frac{d\gamma}{dt}\right) = (D\varphi)_p(\mathrm{v})$$

where $(p, \mathrm{v}) = v(p)$. But by (2.1.6) the term on the right is $L_v\varphi(p)$. Hence we conclude

Theorem 2.2.6. *$\varphi \in C^1(U)$ is an integral of the system (2.2.1) if and only if $L_v\varphi = 0$.*

We'll say that v is *complete* if, for every $p \in U$, there exists an integral curve, $\gamma : \mathbb{R} \to U$ with $\gamma(0) = p$, i.e., for every p there exists an integral curve that starts at p and *exists for all time*. To see what "completeness" involves, we recall that an integral curve

$$\gamma : [0, b) \to U\,,$$

with $\gamma(0) = p$, is called *maximal* if it can't be extended to an interval $[0, b')$, $b' > b$. We claim that for such integral curves either

i. $b = +\infty$
or

ii. $|\gamma(t)| \to +\infty$ as $t \to b$
or

iii. the limit set of

$$\{\gamma(t)\,, \quad 0 \leq t, b\}$$

contains points on the boundary of U.

Proof. Suppose that none of these assertions are true. Then there exists a sequence, $0 < t_i < t$, $i = 1, 2, \ldots$, such that $t_i \to b$ and $\gamma(t_i) \to p_0 \in U$. Let U_0 be a neighborhood of p_0 with the properties described in the existence theorem 2.2.2. Then for i large $\gamma(t_i)$ is in U_0 and $\epsilon = b - t_i < T$. Thus letting $p = \gamma(t_i)$, there exists an integral curve of v,

$$\gamma_p(t)\,, -T + b < t < T + b$$

with $\gamma_p(b) = p$. By reparametrization the curve

$$\gamma_p(t + \epsilon)\,,\ -T + b - \epsilon < t < T + b - \epsilon \tag{2.2.3}$$

is an integral curve of v. Moreover, $\gamma_p(t_i + \epsilon) = \gamma_p(b) = p$, so by the uniqueness theorem 2.2.3 the curve (2.2.9) coincides with $\gamma(t)$ on the interval $-T + b < t < b$ and hence $\gamma(t)$ can be extended to the interval $0 < t < b + T - \epsilon$ by setting it equal to (2.2.9) on $b \leq t < b + T - \epsilon$. This contradicts the maximality of γ and proves the theorem.

□

Hence if we can exclude ii. and iii. we'll have shown that an integral curve with $\gamma(0) = p$ exists for all positive time. A simple criterion for excluding ii. and iii. is the following.

Lemma 2.2.7. *The scenarios ii. and iii. can't happen if there exists a proper C^1-function, $\varphi : U \to \mathbb{R}$ with $L_v\varphi = 0$.*

Proof. $L_v\varphi = 0$ implies that φ is constant on $\gamma(t)$, but if $\varphi(p) = c$ this implies that the curve, $\gamma(t)$, lies on the compact subset, $\varphi^{-1}(c)$, of U; hence it can't "run off to infinity" as in scenario ii. or "run off to the boundary" as in scenario iii.

□

Applying a similar argument to the interval $(-b, 0]$ we conclude:

Theorem 2.2.8. *Suppose there exists a proper C^1-function, $\varphi : U \to \mathbb{R}$ with the property $L_v\varphi = 0$. Then v is complete.*

Example.

Let $U = \mathbb{R}^2$ and let v be the vector field

$$v = x^3 \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} .$$

Then $\varphi(x, y) = 2y^2 + x^4$ is a proper function with the property above.

Another hypothesis on v which excludes ii. and iii. is the following. We'll define the *support* of v to be the closure of the set

$$\{q \in U \, , \quad v(q) \neq 0\} \, ,$$

and will say that v is compactly supported if this set is compact. We will prove

Theorem 2.2.9. *If v is compactly supported it is complete.*

Proof. Notice first that if $v(p) = 0$, the constant curve, $\gamma_0(t) = p$, $-\infty < t < \infty$, satisfies the equation

$$\frac{d}{dt}\gamma_0(t) = 0 = v(p) \, ,$$

so it is an integral curve of v. Hence if $\gamma(t)$, $-a < t < b$, is any integral curve of v with the property, $\gamma(t_0) = p$, for some t_0, it has to coincide with γ_0 on the interval, $-a < t < b$, and hence has to be the constant curve, $\gamma(t) = p$, on this interval.

Now suppose the support, A, of v is compact. Then either $\gamma(t)$ is in A for all t or is in $U - A$ for some t_0. But if this happens, and $p = \gamma(t_0)$ then $v(p) = 0$, so $\gamma(t)$ has to coincide with the constant curve, $\gamma_0(t) = p$, for all t. In neither case can it go off to infinity or off to the boundary of U as $t \to b$.

$\square$

One useful application of this result is the following. Suppose v is a vector field on U, and one wants to see what its integral curves look like on some compact set, $A \subseteq U$. Let $\rho \in \mathcal{C}_0^\infty(U)$ be a bump function which is equal to one on a neighborhood of A. Then the vector field, $w = \rho v$, is compactly supported and hence complete, but it is identical with v on A, so its integral curves on A coincide with the integral curves of v.

If v is complete then for every p, one has an integral curve, $\gamma_p : \mathbb{R} \to U$ with $\gamma_p(0) = p$, so one can define a map

$$f_t : U \to U$$

by setting $f_t(p) = \gamma_p(t)$. If v is $\mathcal{C}^\infty$, this mapping is $\mathcal{C}^\infty$ by the smooth dependence on initial data theorem, and by definition f_0 is the identity map, i.e., $f_0(p) = \gamma_p(0) = p$. We claim that the f_t's also have the property

$$f_t \circ f_a = f_{t+a} \,. \tag{2.2.4}$$

Indeed if $f_a(p) = q$, then by the reparametrization theorem, $\gamma_q(t)$ and $\gamma_p(t+a)$ are both integral curves of v, and since $q = \gamma_q(0) = \gamma_p(a) = f_a(p)$, they have the same initial point, so

$$\begin{aligned} \gamma_q(t) &= f_t(q) = (f_t \circ f_a)(p) \\ &= \gamma_p(t+a) = f_{t+a}(p) \end{aligned}$$

for all t. Since f_0 is the identity it follows from (2.2.2) that $f_t \circ f_{-t}$ is the identity, i.e.,

$$f_{-t} = f_t^{-1} \,,$$

so f_t is a $\mathcal{C}^\infty$ diffeomorphism. Hence if v is complete it generates a "one-parameter group", f_t, $-\infty < t < \infty$, of $\mathcal{C}^\infty$-diffeomorphisms.

For v not complete there is an analogous result, but it's trickier to formulate precisely. Roughly speaking v generates a one-parameter group of diffeomorphisms, f_t, but these diffeomorphisms are not defined on all of U nor for all values of t. Moreover, the identity (2.2.4) only holds on the open subset of U where both sides are well-defined.

We'll devote the remainder of this section to discussing some "functorial" properties of vector fields. Let U and W be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and let $\varphi : U \to W$ be a $\mathcal{C}^\infty$ map. If v is a $\mathcal{C}^\infty$-vector field on U and w a $\mathcal{C}^\infty$-vector field on W we will say that v and w are "φ-related" if, for all $p \in U$ and $q = f(p)$

$$d\varphi_p(v_p) = \mathrm{w}_q \,. \tag{2.2.5}$$

Writing

$$v \;=\; \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}\,, \quad v_i \in \mathcal{C}^\infty(U)$$

and

$$\mathrm{w} \;=\; \sum_{j=1}^m \mathrm{w}_j \frac{\partial}{\partial y_j}\,, \quad \mathrm{w}_j \in \mathcal{C}^\infty(V)$$

this equation reduces, in coordinates, to the equation

$$\mathrm{w}_i(q) = \sum \frac{\partial \varphi_i}{\partial x_j}(p) v_j(p)\,. \tag{2.2.6}$$

In particular, if $m = n$ and φ is a $\mathcal{C}^\infty$ diffeomorphism, the formula (2.2.6) defines a $\mathcal{C}^\infty$-vector field on W, i.e.,

$$\mathrm{w} = \sum_{j=1}^n \mathrm{w}_i \frac{\partial}{\partial y_j}$$

is the vector field defined by the equation

$$\mathrm{w}_i = \sum_{j=1}^n \left(\frac{\partial \varphi_i}{\partial x_j} v_j \right) \circ f^{-1}\,. \tag{2.2.7}$$

Hence we've proved

Theorem 2.2.10. *If $\varphi : U \to W$ is a $\mathcal{C}^\infty$ diffeomorphism and v a $\mathcal{C}^\infty$-vector field on U, there exists a unique $\mathcal{C}^\infty$ vector field, w, on W having the property that v and w are φ-related.*

We'll denote this vector field by $\varphi_* v$ and call it the *push-forward of v by φ*.

We'll leave the following assertions about φ-related vector fields as easy exercises.

Theorem 2.2.11. *Let If v and w are φ-related, every integral curve*

$$\gamma : I \to U_1$$

of v gets mapped by φ onto an integral curve, $\varphi \circ \gamma : I \to U_2$, of w.

Corollary 2.2.12. *Suppose v and w are complete. Let $f_t : U \to U$ $-\infty < t < \infty$, be the one-parameter group of diffeomorphisms generated by v and $g_t : W \to W$ the one-parameter group generated by w. Then $g_t \circ \varphi = \varphi \circ f_t$*

Hints:

1. Theorem 2.2.11 follows from the chain rule: If $p = \gamma(t)$ and $q = \varphi(p)$

$$d\varphi_p \left(\frac{d}{dt} \gamma(t) \right) = \frac{d}{dt} \varphi(\gamma(t)) \,.$$

2. To deduce Corollary 2.2.12 from Theorem 2.2.11 note that for $p \in U$, $f_t(p)$ is just the integral curve, $\gamma_p(t)$ of v with initial point $\gamma_p(0) = p$.

The notion of φ-relatedness can be very succinctly expressed in terms of the Lie differentiation operation. For $f \in \mathcal{C}^\infty(W)$ let $\varphi^* f$ be the composition, $f \circ \varphi$, viewed as a $\mathcal{C}^\infty$ function on U, i.e., for $p \in W$ let $\varphi^* f(p) = f(\varphi(p))$. Then

$$\varphi^* L_w f = L_v \varphi^* f \,. \tag{2.2.8}$$

(To see this note that if $\varphi(p) = q$ then at the point p the right hand side is

$$df_q \circ d\varphi_p =$$

by the chain rule and by definition the left hand side is

$$df_q(w(q)) \,.$$

Moreover, by definition

$$w(q) = d\varphi_p(v(p))$$

so the two sides are the same.)

Another easy consequence of the chain rule is:

Theorem 2.2.13. *Let V be an open subset of $\mathbb{R}^k$, $\psi : W \to V$ a $\mathcal{C}^\infty$ map and u a vector field on V. Then if v and w are φ related and w and u are ψ related, v and u are $\psi \circ \varphi$ related.*

Exercises.

1. Let v be a complete vector field on U and $f_t : U \to U$, the one parameter group of diffeomorphisms generated by v. Show that if $\varphi \in C^1(U)$

$$L_v\varphi = \left(\frac{d}{dt} f_t^* \varphi\right)_{t=0} .$$

2. (a) Let $U = \mathbb{R}^2$ and let $\mathfrak{v}$ be the vector field, $x_1\partial/\partial x_2 - x_2\partial/\partial x_1$. Show that the curve

$$t \in \mathbb{R} \to (r\cos(t+\theta)\,,\, r\sin(t+\theta))$$

is the unique integral curve of $\mathfrak{v}$ passing through the point, $(r\cos\theta, r\sin\theta)$, at $t = 0$.

(b) Let $U = \mathbb{R}^n$ and let $\mathfrak{v}$ be the constant vector field: $\sum c_i\partial/\partial x_i$. Show that the curve

$$t \in \mathbb{R} \to a + t(c_1, \dots, c_n)$$

is the unique integral curve of $\mathfrak{v}$ passing through $a \in \mathbb{R}^n$ at $t = 0$.

(c) Let $U = \mathbb{R}^n$ and let $\mathfrak{v}$ be the vector field, $\sum x_i\partial/\partial x_i$. Show that the curve

$$t \in \mathbb{R} \to e^t(a_1, \dots, a_n)$$

is the unique integral curve of $\mathfrak{v}$ passing through a at $t = 0$.

3. Show that the following are one-parameter groups of diffeomorphisms:

(a) $f_t : \mathbb{R} \to \mathbb{R}\,, \quad f_t(x) = x + t$

(b) $f_t : \mathbb{R} \to \mathbb{R}\,, \quad f_t(x) = e^t x$

(c) $f_t : \mathbb{R}^2 \to \mathbb{R}^2\,, \quad f_t(x, y) = (\cos t\, x - \sin t\, y\,,\, \sin t\, x + \cos t\, y)$

4. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear mapping. Show that the series

$$\exp tA = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

converges and defines a one-parameter group of diffeomorphisms of $\mathbb{R}^n$.

5. (a) What are the infinitesimal generators of the one-parameter groups in exercise 13?

(b) Show that the infinitesimal generator of the one-parameter group in exercise 14 is the vector field

$$\sum a_{i,j} x_j \frac{\partial}{\partial x_i}$$

where $[a_{i,j}]$ is the defining matrix of A.

6. Let U and V be open subsets of $\mathbb{R}^n$ and $f : U \to V$ a diffeomorphism. If w is a vector field on V, define the pull-back, f^*w of w to U to be the vector field

$$f^*w = (f_*^{-1}w)\,.$$

Show that if φ is a $\mathcal{C}^\infty$ function on V

$$f^*L_w\varphi = L_{f^*w}f^*\varphi\,.$$

Hint: (2.2.9).

7. Let U be an open subset of $\mathbb{R}^n$ and v and w vector fields on U. Suppose v is the infinitesimal generator of a one-parameter group of diffeomorphisms

$$f_t : U \to U\,, \quad -\infty < t < \infty\,.$$

Let $w_t = f_t^*w$. Show that for $\varphi \in \mathcal{C}^\infty(U)$

$$L_{[v,w]}\varphi = L_{\dot{w}}\varphi$$

where

$$\dot{w} = \frac{d}{dt}f_t^*w\,|_{t=0}\,.$$

Hint: Differentiate the identity

$$f_t^* L_w \varphi = L_{w_t} f_t^* \varphi$$

with respect to t and show that at $t = 0$ the derivative of the left hand side is

$$L_v L_w \varphi$$

by exercise 1 and the derivative of the right hand side is

$$L_{\dot{w}} + L_w(L_v \varphi)\,.$$

8. Conclude from exercise 7 that

$$[v, w] = \frac{d}{dt} f_t^* w\,|_{t=0}\,. \tag{2.2.9}$$

9. Prove the parametrization Theorem 2.2.2

2.3 k-forms

One-forms are the bottom tier in a pyramid of objects whose k^{th} tier is the space of *k-forms*. More explicitly, given $p \in \mathbb{R}^n$ we can, as in §1.5, form the k^{th} exterior powers

$$\Lambda^k(T_p^*\mathbb{R}^n)\,, \quad k = 1, 2, 3, \dots, n \tag{2.3.1}$$

of the vector space, $T_p^*\mathbb{R}^n$, and since

$$\Lambda^1(T_p^*\mathbb{R}^n) = T_p^*\mathbb{R}^n \tag{2.3.2}$$

one can think of a one-form as a function which takes its value at p in the space (2.3.2). This leads to an obvious generalization.

Definition 2.3.1. *Let U be an open subset of $\mathbb{R}^n$. A k-form, ω, on U is a function which assigns to each point, p, in U an element $\omega(p)$ of the space (2.3.1) .*

The wedge product operation gives us a way to construct lots of examples of such objects.

Example 1.

Let ω_i, $i = 1, \ldots, k$ be one-forms. Then $\omega_1 \wedge \cdots \wedge \omega_k$ is the k-form whose value at p is the wedge product

$$\omega_1(p) \wedge \cdots \wedge \omega_k(p) . \tag{2.3.3}$$

Notice that since $\omega_i(p)$ is in $\Lambda^1(T_p^*\mathbb{R}^n)$ the wedge product (2.3.3) makes sense and is an element of $\Lambda^k(T_p^*\mathbb{R}^n)$.

Example 2.

Let f_i, $i = 1, \ldots, k$ be a real-valued $\mathcal{C}^\infty$ function on U. Letting $\omega_i = df_i$ we get from (2.3.3) a k-form

$$df_1 \wedge \cdots \wedge df_k \tag{2.3.4}$$

whose value at p is the wedge product

$$(df_1)_p \wedge \cdots \wedge (df_k)_p . \tag{2.3.5}$$

Since $(dx_1)_p, \ldots, (dx_n)_p$ are a basis of $T_p^*\mathbb{R}^n$, the wedge products

$$(dx_{i_1})_p \wedge \cdots \wedge (dx_{1_k})_p , \quad 1 \le i_1 < \cdots < i_k \le n \tag{2.3.6}$$

are a basis of $\Lambda^k(T_p^*)$. To keep our multi-index notation from getting out of hand, we'll denote these basis vectors by $(dx_I)_p$, where $I = (i_1, \ldots, i_k)$ and the I's range over multi-indices of length k which are *strictly increasing*. Since these wedge products are a basis of $\Lambda^k(T_p^*\mathbb{R}^n)$ every element of $\Lambda^k(T_p^*\mathbb{R}^n)$ can be written uniquely as a sum

$$\sum c_I (dx_I)_p , \quad c_I \in \mathbb{R}$$

and every k-form, ω, on U can be written uniquely as a sum

$$\omega = \sum f_I \, dx_I \tag{2.3.7}$$

where dx_I is the k-form, $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, and f_I is a real-valued function,

$$f_I : U \to \mathbb{R} .$$

Definition 2.3.2. *The k-form (2.3.7) is of class C^r if each of the f_I's is in $C^r(U)$.*

Henceforth we'll assume, unless otherwise stated, that *all the k-forms we consider are of class $\mathcal{C}^\infty$*, and we'll denote the space of these k-forms by $\Omega^k(U)$.

We will conclude this section by discussing a few simple operations on k-forms.

1. Given a function, $f \in \mathcal{C}^\infty(U)$ and a k-form $\omega \in \Omega^k(U)$ we define $f\omega \in \Omega^k(U)$ to be the k-form

$$p \in U \to f(p)\omega_p \in \Lambda^k(T_p^*\mathbb{R}^n)\,.$$

2. Given $\omega_i \in \Omega^k(U)$, $i = 1, 2$ we define $\omega_1 + \omega_2 \in \Omega^k(U)$ to be the k-form

$$p \in U \to (\omega_1)_p + (\omega_2)_p \in \Lambda^k(T_p^*\mathbb{R}^n)\,.$$

(Notice that this sum makes sense since each summand is in $\Lambda^k(T_p^*\mathbb{R}^n)$.)

3. Given $\omega_1 \in \Omega^{k_1}(U)$ and $\omega_2 \in \Omega^{k_2}(U)$ we define their *wedge product*, $\omega_1 \wedge \omega_2 \in \Omega^{k_1+k_2}(u)$ to be the $(k_1 + k_2)$-form

$$p \in U \to (\omega_1)_p \wedge (\omega_2)_p \in \Lambda^{k_1+k_2}(T_p^*\mathbb{R}^n)\,.$$

We recall that $\Lambda^0(T_p^*\mathbb{R}^n) = \mathbb{R}$, so a zero-form is an $\mathbb{R}$-valued function and a zero form of class $\mathcal{C}^\infty$ is a $\mathcal{C}^\infty$ function, i.e.,

$$\Omega^0(U) = \mathcal{C}^\infty(U)\,.$$

A fundamental operation on forms is the "d-operation" which associates to a function $f \in \mathcal{C}^\infty(U)$ the 1-form df. It's clear from the identity (2.1.9) that df is a 1-form of class $\mathcal{C}^\infty$, so the d-operation can be viewed as a map

$$d : \Omega^0(U) \to \Omega^1(U)\,. \tag{2.3.8}$$

We will show in the next section that an analogue of this map exists for every $\Omega^k(U)$.

Exercises.

1. Let $\omega \in \Omega^2(\mathbb{R}^4)$ be the 2-form, $dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. Compute $\omega \wedge \omega$.

2. Let $\omega_i \in \Omega^1(\mathbb{R}^3)$, $i = 1, 2, 3$ be the 1-forms

$$\begin{aligned} \omega_1 &= x_2\,dx_3 - x_3\,dx_2 \\ \omega_2 &= x_3\,dx_1 - x_1\,dx_3 \end{aligned}$$

and

$$\omega_3 = x_1\,dx_2 - x_2\,dx_1\,.$$

Compute

(a) $\omega_1 \wedge \omega_2$.

(b) $\omega_2 \wedge \omega_3$.

(c) $\omega_3 \wedge \omega_1$.

(d) $\omega_1 \wedge \omega_2 \wedge \omega_3$.

3. Let U be an open subset of $\mathbb{R}^n$ and $f_i \in \mathcal{C}^\infty(U)$, $i = 1, \ldots, n$. Show that

$$df_1 \wedge \cdots \wedge df_n = \det \left[\frac{\partial f_i}{\partial x_j} \right] dx_1 \wedge \cdots \wedge dx_n \,.$$

4. Let U be an open subset of $\mathbb{R}^n$. Show that every $(n-1)$-form, $\omega \in \Omega^{n-1}(U)$, can be written uniquely as a sum

$$\sum_{i=1}^{n} f_i \; dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

where $f_i \in \mathcal{C}^\infty(U)$ and the "cap" over dx_i means that dx_i is to be deleted from the product, $dx_1 \wedge \cdots \wedge dx_n$.

5. Let $\mu = \sum_{i=1}^{n} x_i dx_i$. Show that there exists an $(n-1)$-form, $\omega \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$ with the property

$$\mu \wedge \omega = dx_1 \wedge \cdots \wedge dx_n \,.$$

6. Let J be the multi-index $(j_1, \ldots, j_k)$ and let $dx_J = dx_{j_1} \wedge \cdots \wedge dx_{j_k}$. Show that $dx_J = 0$ if $j_r = j_s$ for some $r \neq s$ and show that if the j_r's are all distinct

$$dx_J = (-1)^\sigma \, dx_I$$

where $I = (i_1, \ldots, i_k)$ is the strictly increasing rearrangement of $(j_1, \ldots, j_k)$ and σ is the permutation

$$j_1 \to i_1, \ldots, j_k \to i_k \,.$$

7. Let I be a strictly increasing multi-index of length k and J a strictly increasing multi-index of length ℓ. What can one say about the wedge product $dx_I \wedge dx_J$?

2.4 Exterior differentiation

Let U be an open subset of $\mathbb{R}^n$. In this section we are going to define an operation

$$d : \Omega^k(U) \to \Omega^{k+1}(U)\,. \tag{2.4.1}$$

This operation is called *exterior differentiation* and is the fundamental operation in n-dimensional vector calculus.

For $k = 0$ we already defined the operation (2.4.1) in §2.1.1.. Before defining it for the higher k's we list some properties that we will require to this operation to satisfy.

Property I. For ω_1 and ω_2 in $\Omega^k(U)$, $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$.

Property II. For $\omega_1 \in \Omega^k(U)$ and $\omega_2 \in \Omega^\ell(U)$

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2\,. \tag{2.4.2}$$

Property III. For $\omega \in \Omega^k(U)$

$$d(d\omega) = 0\,. \tag{2.4.3}$$

Let's point out a few consequences of these properties. First note that by Property III

$$d(df) = 0 \tag{2.4.4}$$

for every function, $f \in \mathcal{C}^\infty(U)$. More generally, given k functions, $f_i \in \mathcal{C}^\infty(U)$, $i = 1, \dots, k$, then by combining (2.4.4) with (2.4.2) we get by induction on k:

$$d(df_1 \wedge \cdots \wedge df_k) = 0\,. \tag{2.4.5}$$

Proof. Let $\mu = df_2 \wedge \cdots \wedge df_k$. Then by induction on k, $d\mu = 0$; and hence by (2.4.2) and (2.4.4)

$$d(df_1 \wedge \mu) = d(d_1 f) \wedge \mu + (-1)\, df_1 \wedge d\mu = 0\,,$$

as claimed.)

In particular, given a multi-index, $I = (i_1, \dots, i_k)$ with $1 \leq i_r \leq n$

$$d(dx_I) = d(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = 0 \,. \tag{2.4.6}$$

Recall now that every k-form, $\omega \in \Omega^k(U)$, can be written uniquely as a sum

$$\omega = \sum f_I \, dx_I \,, \quad f_I \in \mathcal{C}^\infty(U)$$

where the multi-indices, I, are strictly increasing. Thus by (2.4.2) and (2.4.6)

$$d\omega = \sum df_I \wedge dx_I \,. \tag{2.4.7}$$

This shows that if there exists a "d" with properties I—III, it has to be given by the formula (2.4.7). Hence all we have to show is that the operator defined by this formula has these properties. Property I is obvious. To verify Property II we first note that for I strictly increasing (2.4.6) is a special case of (2.4.7). (Take $f_I = 1$ and $f_J = 0$ for $J \neq I$.) Moreover, if I is not strictly increasing it is either repeating, in which case $dx_I = 0$, or non-repeating in which case I^σ is strictly increasing for some permutation, $\sigma \in S_k$, and

$$dx_I = (-1)^\sigma \, dx_{I^\sigma} \,. \tag{2.4.8}$$

Hence (2.4.7) implies (2.4.6) for *all* multi-indices I. The same argument shows that for *any* sum over indices, I, for length k

$$\sum f_I dx_I$$

one has the identity:

$$d(\sum f_I \, dx_I) = \sum df_I \wedge dx_I \,. \tag{2.4.9}$$

(As above we can ignore the repeating I's, since for these I's, $dx_I = 0$, and by (2.4.8) we can make the non-repeating I's strictly increasing.)

Suppose now that $\omega_1 \in \Omega^k(U)$ and $\omega_2 \in \Omega^\ell(U)$. Writing

$$\omega_1 = \sum f_I \, dx_I$$

and

$$\omega_2 = \sum g_J \, dx_J$$

with f_I and g_J in $\mathcal{C}^\infty(U)$ we get for the wedge product

$$\omega_1 \wedge \omega_2 = \sum f_I g_J \, dx_I \wedge dx_J \tag{2.4.10}$$

and by (2.4.9)

$$d(\omega_1 \wedge \omega_2) = \sum d(f_I g_J) \wedge dx_I \wedge dx_J \,. \tag{2.4.11}$$

(Notice that if $I = (i_1, \cdots, i_k)$ and $J = (j_i, \ldots, i_\ell)$, $dx_I \wedge dx_J = dx_K$, K being the multi-index, $(i_1, \ldots, i_k, j_1, \ldots, j_\ell)$. Even if I and J are strictly increasing, K won't necessarily be strictly increasing. However in deducing (2.4.11) from (2.4.10) we've observed that this doesn't matter .) Now note that by (2.1.10)

$$d(f_I g_J) = g_J \, df_I + f_I \, dg_J \,,$$

and by the wedge product identities of §(**??**),

$$\begin{aligned} dg_J \wedge dx_I &= dg_J \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= (-1)^k \, dx_I \wedge dg_J \,, \end{aligned}$$

so the sum (2.4.11) can be rewritten:

$$\sum df_I \wedge dx_I \wedge g_J \, dx_J + (-1)^k \sum f_I \, dx_I \wedge dg_J \wedge dx_J \,,$$

or

$$\left(\sum df_I \wedge dx_I\right) \wedge \left(\sum g_J \, dx_J\right) + (-1)^k \left(\sum dg_J \wedge dx_J\right) \,,$$

or finally:

$$d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2 \,.$$

Thus the "d" defined by (2.4.7) has Property II. Let's now check that it has Property III. If $\omega = \sum f_I \, dx_I$, $f_I \in \mathcal{C}^\infty(U)$, then by definition, $d\omega = \sum df_I \wedge dx_I$ and by (2.4.6) and (2.4.2)

$$d(d\omega) = \sum d(df_I) \wedge dx_I \,,$$

so it suffices to check that $d(df_I) = 0$, i.e., it suffices to check (2.4.4) for zero forms, $f \in \mathcal{C}^\infty(U)$. However, by (2.1.9)

$$df = \sum_{j=1}^n \frac{\partial f}{\partial x_j} \, dx_j$$

so by (2.4.7)

$$\begin{aligned} d(df) &= \sum_{j=1}^{n} d\left(\frac{\partial f}{\partial x_j}\right) dx_j \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}\, dx_i\right) \wedge dx_j \\ &= \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}\, dx_i \wedge dx_j\,. \end{aligned}$$

Notice, however, that in this sum, $dx_i \wedge dx_j = -dx_j \wedge dx_i$ and

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

so the (i,j) term cancels the (j,i) term, and the total sum is zero.

□

A form, $\omega \in \Omega^k(U)$, is said to be *closed* if $d\omega = 0$ and is said to be *exact* if $\omega = d\mu$ for some $\mu \in \Omega^{k-1}(U)$. By Property III every exact form is closed, but the converse is not true even for 1-forms. (See §2.1.1., exercise 8). In fact it's a very interesting (and hard) question to determine if an open set, U, has the property: "For $k > 0$ every closed k-form is exact."[1]

Some examples of sets with this property are described in the exercises at the end of §2.5. We will also sketch below a proof of the following result (and ask you to fill in the details).

Lemma 2.4.1 (Poincaré's Lemma.). *If ω is a closed form on U of degree $k > 0$, then for every point, $p \in U$, there exists a neighborhood of p on which ω is exact.*

(See exercises 5 and 6 below.)

Exercises:

1. Compute the exterior derivatives of the forms below.

[1]For $k = 0$, $df = 0$ *doesn't* imply that f is exact. In fact "exactness" doesn't make much sense for zero forms since there aren't any "-1" forms. However, if $f \in \mathcal{C}^\infty(U)$ and $df = 0$ then f is constant on connected components of U. (See § 2.1.1., exercise 2.)

(a) $x_1\,dx_2 \wedge dx_3$

(b) $x_1\,dx_2 - x_2\,dx_1$

(c) $e^{-f}\,df$ where $f = \sum_{i=1}^n x_i^2$

(d) $\sum_{i=1}^n x_i\,dx_i$

(e) $\sum_{i=1}^n (-1)^i x_i\,dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$

2. Solve the equation: $d\mu = \omega$ for $\mu \in \Omega^1(\mathbb{R}^3)$, where ω is the 2-form

(a) $dx_2 \wedge\ dx_3$

(b) $x_2\,dx_2 \wedge\ dx_3$

(c) $(x_1^2 + x_2^2)\,dx_1 \wedge\ dx_2$

(d) $\cos x_1\,dx_1 \wedge\ dx_3$

3. Let U be an open subset of $\mathbb{R}^n$.

(a) Show that if $\mu \in \Omega^k(U)$ is exact and $\omega \in \Omega^\ell(U)$ is closed then $\mu \wedge \omega$ is exact. *Hint:* The formula (2.4.2).

(b) In particular, dx_1 is exact, so if $\omega \in \Omega^\ell(U)$ is closed $dx_1 \wedge \omega = d\mu$. What is μ?

4. Let Q be the rectangle, $(a_1, b_1) \times \cdots \times (a_n, b_n)$. Show that if ω is in $\Omega^n(Q)$, then ω is exact.

Hint: Let $\omega = f\,dx_1 \wedge \cdots \wedge dx_n$ with $f \in \mathcal{C}^\infty(Q)$ and let g be the function

$$g(x_1, \ldots, x_n) = \int_{a_1}^{x_1} f(t, x_2, \ldots, x_n)\,dt\,.$$

Show that $\omega = d(g\,dx_2 \wedge \cdots \wedge dx_n)$.

5. Let U be an open subset of $\mathbb{R}^{n-1}$, $A \subseteq \mathbb{R}$ an open interval and (x,t) product coordinates on $U \times A$. We will say that a form, $\mu \in \Omega^\ell(U \times A)$ is *reduced* if it can be written as a sum

$$\mu = \sum f_I(x,t)\,dx_I\,, \tag{2.4.12}$$

(i.e., no terms involving dt).

(a) Show that every form, $\omega \in \Omega^k(U \times A)$ can be written uniquely as a sum:

$$\omega = dt \wedge \alpha + \beta \tag{2.4.13}$$

where α and β are reduced.

(b) Let μ be the reduced form (2.4.12) and let

$$\frac{d\mu}{dt} = \sum \frac{d}{dt} f_I(x,t)\, dx_I$$

and

$$d_U \mu = \sum_I \left(\sum_{i=1}^n \frac{\partial}{\partial x_i} f_I(x,t)\, dx_i \right) \wedge\, dx_I\,.$$

Show that

$$d\mu = \, dt \wedge \frac{d\mu}{dt} + d_U \mu\,.$$

(c) Let ω be the form (2.4.13). Show that

$$d\omega = dt \wedge\, d_U \alpha + \, dt \wedge \frac{d\beta}{dt} + \, d_U \beta$$

and conclude that ω is closed if and only if

$$\begin{aligned} \frac{d\beta}{dt} &= d_U \alpha \\ d\beta_U &= 0\,. \end{aligned} \tag{2.4.14}$$

(d) Let α be a reduced $(k-1)$-form. Show that there exists a reduced $(k-1)$-form, ν, such that

$$\frac{d\nu}{dt} = \alpha\,. \tag{2.4.15}$$

Hint: Let $\alpha = \sum f_I(x,t)\, dx_I$ and $\nu = \sum g_I(x,t)\, dx_I$. The equation (2.4.15) reduces to the system of equations

$$\frac{d}{dt}\, g_I(x,t) = f_I(x,t)\,. \tag{2.4.16}$$

Let c be a point on the interval, A, and using freshman calculus show that (2.4.16) has a unique solution, $g_I(x,t)$, with $g_I(x,c) = 0$.

(e) Show that if ω is the form (2.4.13) and ν a solution of (2.4.15) then the form

$$\omega - d\nu \tag{2.4.17}$$

is reduced.

(f) Let

$$\gamma = \sum h_I(x,t)\, dx)I$$

be a reduced k-form. Deduce from (2.4.14) that if γ is closed then $\frac{d\gamma}{dt} = 0$ and $d_U\gamma = 0$. Conclude that $h_I(x,t) = h_I(x)$ and that

$$\gamma = \sum h_I(x)\, dx_I$$

is effectively a closed k-form on U. Now prove: If every closed k-form on U is exact, then every closed k-form on $U \times A$ is exact. *Hint:* Let ω be a closed k-form on $U \times A$ and let γ be the form (2.4.17).

6. Let $Q \subseteq \mathbb{R}^n$ be an open rectangle. Show that every closed form on Q of degree $k > 0$ is exact. *Hint:* Let $Q = (a_1, b_1) \times \cdots \times (a_n, b_n)$. Prove this assertion by induction, at the n^{th} stage of the induction letting $U = (a_1, b_1) \times \cdots \times (a_{n-1}, b_{n-1})$ and $A = (a_n, b_n)$.

2.5 The interior product operation

In §2.1.1. we explained how to pair a one-form, ω, and a vector field, v, to get a function, $\iota(v)\omega$. This pairing operation generalizes: If one is given a k-form, ω, and a vector field, v, both defined on an open subset, U, one can define a $(k-1)$-form on U by defining its value at $p \in U$ to be the interior product

$$\iota(v(p))\omega(p)\,. \tag{2.5.1}$$

Note that $v(p)$ is in $T_p\mathbb{R}^n$ and $\omega(p)$ in $\Lambda^k(T_p^*\mathbb{R}^n)$, so by definition of interior product (see §1.7), the expression (2.5.1) is an element of $\Lambda^{k-1}(T_p^*\mathbb{R}^n)$. We will denote by $\iota(v)\omega$ the $(k-1)-$form on U whose value at p is (2.5.1). From the properties of interior product on vector spaces which we discussed in §1.7, one gets analogous properties for this interior product on forms. We will list these properties, leaving their verification as an exercise. Let v and ω be vector fields, and ω_1

and ω_2 k-forms, ω a k-form and μ an ℓ-form. Then $\iota(v)\omega$ is linear in ω:

$$\iota(v)(\omega_1 + \omega_2) = \iota(v)\omega_1 + \iota(v)\omega_2 \,, \tag{2.5.2}$$

linear in v:

$$\iota(v + w)\omega = \iota(v)\omega + z(w)\omega \,, \tag{2.5.3}$$

has the derivation property:

$$\iota(v)(\omega \wedge \mu) = \iota(v)\omega \wedge \mu + (-1)^k \omega \wedge \iota(v)\mu \tag{2.5.4}$$

satisfies the identity

$$\iota(v)(\iota(w)\omega) = -\iota(w)(\iota(v)\omega) \tag{2.5.5}$$

and, as a special case of (2.5.5), the identity,

$$\iota(v)(\iota(v)\omega) = 0 \,. \tag{2.5.6}$$

Moreover, if ω is "decomposable" i.e., is a wedge product of one-forms

$$\omega \quad = \quad \mu_1 \wedge \cdots \wedge \mu_k \,, \tag{2.5.7}$$

then

$$\iota(v)\omega \quad = \quad \sum_{r=1}^{k} (-1)^{r-1} (\iota(v)\mu_r)\mu_1 \wedge \cdots \widehat{\mu}_r \cdots \wedge \mu_k \,. \tag{2.5.8}$$

We will also leave for you to prove the following two assertions, both of which are special cases of (2.5.8). If $v = \partial/\partial x_r$ and $\omega = dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ then

$$\iota(v)\omega \quad = \quad \sum_{r=1}^{k} (-1)^r \delta^i_{i_r} \, dx_{I_r} \tag{2.5.9}$$

where

$$\delta^i_{i_r} \quad = \quad \begin{cases} 1 & i = i_r \\ 0 \,, & i \neq i_r \end{cases} \,.$$

and $I_r = (i_1, \ldots, \widehat{i}_r, \ldots, i_k)$ and if $v = \sum f_i \, \partial/\partial x_i$ and $\omega = dx_1 \wedge \cdots \wedge dx_n$ then

$$\iota(v)\omega = \sum (-1)^{r-1} f_r \, dx_1 \wedge \cdots \widehat{dx}_r \cdots \wedge dx_n \,. \tag{2.5.10}$$

By combining exterior differentiation with the interior product operation one gets another basic operation of vector fields on forms: the *Lie differentiation* operation. For zero-forms, i.e., for $\mathcal{C}^\infty$ functions, φ, we defined this operation by the formula (2.1.13). For k-forms we'll define it by the slightly more complicated formula

$$L_v \omega = \iota(v) \, d\omega + \, d\iota(v)\omega \,. \tag{2.5.11}$$

(Notice that for zero-forms the second summand is zero, so (2.5.11) and (2.1.13) agree.) If ω is a k-form the right hand side of (2.5.11) is as well, so L_v takes k-forms to k-forms. It also has the property

$$dL_v \omega = L_v \, d\omega \tag{2.5.12}$$

i.e., it "commutes" with d, and the property

$$L_v(\omega \wedge \mu) = L_v \omega \wedge \mu + \omega \wedge L_v \mu \tag{2.5.13}$$

and from these properties it is fairly easy to get an explicit formula for $L_v \omega$. Namely let ω be the k-form

$$\omega = \sum f_I \, dx_I \,, \quad f_I \in \mathcal{C}^\infty(U)$$

and v the vector field

$$\sum g_i \, \partial/\partial x_i \,, \quad g_i \in \mathcal{C}^\infty(U) \,.$$

By (2.5.13)

$$L_v(f_I \, dx_I) \quad = \quad (L_v f_I) \, dx_I + f_I (L_v \, dx_I)$$

and

$$L_v \, dx_I \quad = \quad \sum_{r=1}^{k} dx_{i_1} \wedge \cdots \wedge L_v \, dx_{i_r} \wedge \cdots \wedge dx_{i_k} \,,$$

and by (2.5.12)

$$L_v \, dx_{i_r} = dL_v x_{i_r}$$

so to compute $L_v\omega$ one is reduced to computing $L_v x_{i_r}$ and $L_v f_I$. However by (2.5.13)

$$L_v x_{i_r} = g_{i_r}$$

and

$$L_v f_I = \sum g_i \frac{\partial f_I}{\partial x_i} .$$

We will leave the verification of (2.5.12) and (2.5.13) as exercises, and also ask you to prove (by the method of computation that we've just sketched) the *divergence formula*

$$L_v(dx_1 \wedge \cdots \wedge dx_n) = \sum \left(\frac{\partial g_i}{\partial x_i} \right) dx_1 \wedge \cdots \wedge dx_n . \tag{2.5.14}$$

Exercises:

1. Verify the assertions (2.5.2)—(2.5.7).

2. Show that if ω is the k-form, dx_I and v the vector field, $\partial/\partial x_r$, then $\iota(v)\omega$ is given by (2.5.9).

3. Show that if ω is the n-form, $dx_1 \wedge \cdots \wedge dx_n$, and v the vector field, $\sum f_i \, \partial/\partial x_i$, $\iota(v)\omega$ is given by (2.5.10).

4. Let U be an open subset of $\mathbb{R}^n$ and v a $\mathcal{C}^\infty$ vector field on U. Show that for $\omega \in \Omega^k(U)$

$$dL_v\omega = L_v \, d\omega$$

and

$$\iota_v L_v \omega = L_v \iota_v \omega .$$

Hint: Deduce the first of these identities from the identity $d(d\omega) = 0$ and the second from the identity $\iota(v)(\iota(v)\omega) = 0$.)

5. Given $\omega_i \in \Omega^{k_i}(U)$, $i = 1, 2$, show that

$$L_v(\omega_1 \wedge \omega_2) = L_v\omega_1 \wedge \omega_2 + \omega_1 \wedge L_v\omega_2 .$$

Hint: Plug $\omega = \omega_1 \wedge \omega_2$ into (2.5.11) and use (2.4.2) and (2.5.4)to evaluate the resulting expression.

6. Let v_1 and v_2 be vector fields on U and let w be their Lie bracket. Show that for $\omega \in \Omega^k(U)$

$$L_w \omega = L_{v_1}(L_{v_2}\omega) - L_{v_2}(L_{v_1}\omega)\,.$$

Hint: By definition this is true for zero-forms and by (2.5.12) for exact one-forms. Now use the fact that every form is a sum of wedge products of zero-forms and one-forms and the fact that L_v satisfies the product identity (2.5.13).

7. Prove the divergence formula (2.5.14).

8. (a) Let $\omega = \Omega^k(\mathbb{R}^n)$ be the form

$$\omega = \sum f_I(x_1, \dots, x_n)\, dx_I$$

and $\mathfrak{v}$ the vector field, $\partial/\partial x_n$. Show that

$$L_{\mathfrak{v}}\omega = \sum \frac{\partial}{\partial x_n} f_I(x_1, \dots, x_n)\, dx_I\,.$$

(b) Suppose $\iota(\mathfrak{v})\omega = L_{\mathfrak{v}}\omega = 0$. Show that ω only depends on $x_1, \dots, x_{k-1}$ and $dx_1, \dots, dx_{k-1}$, i.e., is effectively a k-form on $\mathbb{R}^{n-1}$.

(c) Suppose $\iota(\mathfrak{v})\omega = d\omega = 0$. Show that ω is effectively a closed k-form on $\mathbb{R}^{n-1}$.

(d) Use these results to give another proof of the Poincaré lemma for $\mathbb{R}^n$. Prove by induction on n that every closed form on $\mathbb{R}^n$ is exact.

Hints:

i. Let ω be the form in part (a) and let

$$g_I(x_1, \dots, x_n) = \int_0^{x_n} f_I(x_1, \dots, x_{n-1}, t)\, dt\,.$$

Show that if $\nu = \sum g_I\, dx_I$, then $L_{\mathfrak{v}}\nu = \omega$.

ii. Conclude that

$$(*) \qquad \omega - d\iota(\mathfrak{v})\nu = \iota(\mathfrak{v})\, d\nu\,.$$

iii. Suppose $d\omega = 0$. Conclude from (*) and from the formula (2.5.6) that the form $\beta = \iota(\mathfrak{v})\, d\nu$ satisfies $d\beta = \iota(\mathfrak{v})\beta = 0$.

iv. By part c, β is effectively a closed form on $\mathbb{R}^{n-1}$, and by induction, $\beta = d\alpha$. Thus by (*)

$$\omega = d\iota(\mathfrak{v})\nu + d\alpha\,.$$

2.6 The pull-back operation on forms

Let U be an open subset of $\mathbb{R}^n$, V an open subset of $\mathbb{R}^m$ and $f : U \to V$ a $\mathcal{C}^\infty$ map. Then for $p \in U$ and $q = f(p)$, the derivative of f at p

$$df_p : T_p\mathbb{R}^n \to T_q\mathbb{R}^m$$

is a linear map, so (as explained in §7 of Chapter **??**) one gets from it a pull-back map

$$df_p^* : \Lambda^k(T_q^*\mathbb{R}^m) \to \Lambda^k(T_p^*\mathbb{R}^n)\,. \tag{2.6.1}$$

In particular, let ω be a k-form on V. Then at $q \in V$, ω takes the value

$$\omega_q \in \Lambda^k(T_q^*\mathbb{R}^m)\,,$$

so we can apply to it the operation (2.7.1), and this gives us an element:

$$df_p^*\omega_q \in \Lambda^k(T_p^*\mathbb{R}^n)\,. \tag{2.6.2}$$

In fact we can do this for every point $p \in U$, so this gives us a function,

$$p \in U \to (df_p)^*\omega_q\,, \quad q = f(p)\,. \tag{2.6.3}$$

By the definition of k-form such a function is a k-form on U. We will denote this k-form by $f^*\omega$ and define it to be the *pull-back of* ω *by the map* f. A few of its basic properties are described below.

1. Let φ be a zero-form, i.e., a function, $\varphi \in \mathcal{C}^\infty(V)$. Since

$$\Lambda^0(T_p^*) = \Lambda^0(T_q^*) = \mathbb{R}$$

the map (2.7.1) is just the identity map of $\mathbb{R}$ onto $\mathbb{R}$ when k is equal to zero. Hence for zero-forms

$$(f^*\varphi)(p) = \varphi(q)\,, \tag{2.6.4}$$

i.e., $f^*\varphi$ is just the composite function, $\varphi \circ f \in \mathcal{C}^\infty(U)$.

2. Let $\mu \in \Omega^1(V)$ be the 1-form, $\mu = d\varphi$. By the chain rule (2.6.2) unwinds to:

$$(df_p)^*d\varphi_q = (d\varphi)_q \circ df_p = d(\varphi \circ f)_p \tag{2.6.5}$$

and hence by (2.6.4)

$$f^*\,d\varphi = df^*\varphi\,. \tag{2.6.6}$$

3. If ω_1 and ω_2 are in $\Omega^k(V)$ we get from (2.6.2)

$$(df_p)^*(\omega_1+\omega_2)_q = (df_p)^*(\omega_1)_q + (df_p)^*(\omega_2)_q\,,$$

and hence by (2.6.3)

$$f^*(\omega_1+\omega_2) = f^*\omega_1 + f^*\omega_2\,.$$

4. We observed in § **??** that the operation (2.7.1) commutes with wedge-product, hence if ω_1 is in $\Omega^k(V)$ and ω_2 is in $\Omega^\ell(V)$

$$df_p^*(\omega_1)_q \wedge (\omega_2)_q = df_p^*(\omega_1)_q \wedge df_p^*(\omega_2)_q\,.$$

In other words

$$f^*\omega_1\wedge\omega_2 = f^*\omega_1 \wedge f^*\omega_2\,. \tag{2.6.7}$$

5. Let W be an open subset of $\mathbb{R}^k$ and $g : V \to W$ a $\mathcal{C}^\infty$ map. Given a point $p \in U$, let $q = f(p)$ and $w = g(q)$. Then the composition of the map

$$(df_p)^* : \Lambda^k(T_q^*) \to \Lambda^k(T_p^*)$$

and the map

$$(dg_q)^* : \Lambda^k(T_w^*) \to \Lambda^k(T_q^*)$$

is the map

$$(dg_q \circ df_p)^* : \Lambda^k(T_w^*) \to \Lambda^k(T_p^*)$$

by formula (**??**) of Chapter 1. However, by the chain rule

$$(dg_q)\circ(df)_p = d(g\circ f)_p$$

so this composition is the map

$$d(g\circ f)_p^* : \Lambda^k(T_w^*) \to \Lambda^k(T_p^*)\,.$$

Thus if ω is in $\Omega^k(W)$

$$f^*(g^*\omega) = (g\circ f)^*\omega\,. \tag{2.6.8}$$

Let's see what the pull-back operation looks like in coordinates. Using multi-index notation we can express every k-form, $\omega \in \Omega^k(V)$ as a sum over multi-indices of length k

$$\omega = \sum \varphi_I\, dx_I\,, \tag{2.6.9}$$

the coefficient, φ_I, of dx_I being in $\mathcal{C}^\infty(V)$. Hence by (2.6.4)

$$f^*\omega = \sum f^*\varphi_I f^*(dx_I)$$

where $f^*\varphi_I$ is the function of $\varphi \circ f$. What about $f^*\, dx_I$? If I is the multi-index, $(i_1, \dots, i_k)$, then by definition

$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

so

$$d^*\, dx_I = f^*\, dx_i \wedge \cdots \wedge f^*\, dx_{i_k}$$

by (2.6.7), and by (2.6.6)

$$f^*\, dx_i = df^* x_i = df_i$$

where f_i is the i^{th} coordinate function of the map f. Thus, setting

$$df_I = df_{i_1} \wedge \cdots \wedge df_{i_k}\,,$$

we get for each multi-index, I,

(2.6.10) $$f^*\, dx_I = df_I$$

and for the pull-back of the form (2.6.9)

(2.6.11) $$f^*\omega = \sum f^*\varphi_I\, df_I\,.$$

We will use this formula to prove that pull-back commutes with exterior differentiation:

(2.6.12) $$d\, f^*\omega = f^*\, d\omega\,.$$

To prove this we recall that by (2.3.5), $d(\,df_I) = 0$, hence by (2.3.2) and (2.6.10)

$$\begin{aligned} d\, f^*\omega &= \sum d\, f^*\varphi_I \wedge df_I \\ &= \sum f^*\, d\varphi_I \wedge df^*\, dx_I \\ &= f^* \sum d\varphi_I \wedge dx_I \\ &= f^*\, d\omega\,. \end{aligned}$$

A special case of formula (2.6.10) will be needed in Chapter 4: Let U and V be open subsets of $\mathbb{R}^n$ and let $\omega = dx_1 \wedge \cdots \wedge dx_n$. Then by (2.6.10)

$$f^*\omega_p = (df_1)_p \wedge \cdots \wedge (df_n)_p$$

for all $p \in U$. However,

$$(df_i)_p = \sum \frac{\partial f_i}{\partial x_j}(p)(dx_j)_p$$

and hence by formula (**??**) of Chapter **??**

$$f^*\omega_p = \det\left[\frac{\partial f_i}{\partial x_j}(p)\right](dx_1 \wedge \cdots \wedge dx_n)_p\,.$$

In other words

$$f^*\, dx_1 \wedge \cdots \wedge dx_n = \det\left[\frac{\partial f_i}{\partial x_j}\right] dx_1 \wedge \cdots \wedge dx_n\,. \tag{2.6.13}$$

We will outline in exercises 4 and 5 below the proof of an important topological property of the pull-back operation. Let U be an open subset of $\mathbb{R}^n$, V an open subset of $\mathbb{R}^m$, $A \subseteq \mathbb{R}$ an open interval containing 0 and 1 and $f_i : U \to V$, $i = 0, 1$, a $\mathcal{C}^\infty$ map.

Definition 2.6.1. *A $\mathcal{C}^\infty$ map, $F : U \times A \to V$, is a homotopy between f_0 and f_1 if $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$.*

Thus, intuitively, f_0 and f_1 are *homotopic* if there exists a family of $\mathcal{C}^\infty$ maps, $f_t : U \to V$, $f_t(x) = F(x,t)$, which "smoothly deform f_0 into f_1". In the exercises mentioned above you will be asked to verify that for f_0 and f_1 to be homotopic they have to satisfy the following criteria.

Theorem 2.6.2. *If f_0 and f_1 are homotopic then for every closed form, $\omega \in \Omega^k(V)$, $f_1^*\omega - f_0^*\omega$ is exact.*

This theorem is closely related to the Poincaré lemma, and, in fact, one gets from it a slightly stronger version of the Poincaré lemma than that described in exercises 5–6 in §2.3.

Definition 2.6.3. *An open subset, U, of $\mathbb{R}^n$ is* contractable *if, for some point $p_0 \in U$, the identity map*

$$f_1 : U \to U\,, \quad f(p) = p\,,$$

is homotopic to the constant map

$$f_0 : U \to U\,, \quad f_0(p) = p_0\,.$$

From the theorem above it's easy to see that the Poincaré lemma holds for contractable open subsets of $\mathbb{R}^n$. If U is contractable every closed k-form on U of degree $k > 0$ is exact. (Proof: Let ω be such a form. Then for the identity map $f_0^*\omega = \omega$ and for the constant map, $f_0^*\omega = 0$.)

Exercises.

1. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the map

$$f(x_1, x_2, x_3) = (x_1x_2, x_2x_3^2, x_3^3)\,.$$

Compute the pull-back, $f^*\omega$ for

(a) $\omega = x_2\, dx_3$

(b) $\omega = x_1\, dx_1 \wedge dx_3$

(c) $\omega = x_1\, dx_1 \wedge dx_2 \wedge dx_3$

2. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the map

$$f(x_1, x_2) = (x_1^2, x_2^2, x_1x_2)\,.$$

Complete the pull-back, $f^*\omega$, for

(a) $\omega = x_2\, dx_2 + x_3\, dx_3$

(b) $\omega = x_1\, dx_2 \wedge dx_3$

(c) $\omega = dx_1 \wedge dx_2 \wedge dx_3$

3. Let U be an open subset of $\mathbb{R}^n$, V an open subset of $\mathbb{R}^m$, $f : U \to V$ a $\mathcal{C}^\infty$ map and $\gamma : [a, b] \to U$ a $\mathcal{C}^\infty$ curve. Show that for $\omega \in \Omega^1(V)$

$$\int_\gamma f^*\omega = \int_{\gamma_1} \omega$$

where $\gamma_1 : [a, b] \to V$ is the curve, $\gamma_1(t) = f(\gamma(t))$. (See § 2.1.1., exercise 7.)

4. Let U be an open subset of $\mathbb{R}^n$, $A \subseteq \mathbb{R}$ an open interval containing the points, 0 and 1, and (x,t) product coordinates on $U \times A$. Recall (§ 2.3, exercise 5) that a form, $\mu \in \Omega^\ell(U \times A)$ is *reduced* if it can be written as a sum

$$\mu = \sum f_I(x,t)\, dx_I \tag{2.6.14}$$

(i.e., none of the summands involve "dt"). For a reduced form, μ, let $Q\mu \in \Omega^\ell(U)$ be the form

$$Q\mu = \left(\sum \int_0^1 f_I(x,t)\, dt \right) dx_I \tag{2.6.15}$$

and let $\mu_i \in \Omega^\ell(U)$, $i = 0, 1$ be the forms

$$\mu_0 \;=\; \sum f_I(x,0)\, dx_I \tag{2.6.16}$$

and

$$\mu_1 \;=\; \sum f_I(x,1)\, dx_I\,. \tag{2.6.17}$$

Now recall that every form, $\omega \in \Omega^k(U \times A)$ can be written uniquely as a sum

$$\omega = dt \wedge \alpha + \beta \tag{2.6.18}$$

where α and β are reduced. (See exercise 5 of § 2.4, part a.)

(a) Prove

Theorem 2.6.4. *If the form (2.6.18) is closed then*

$$\beta_0 - \beta_1 = dQ\alpha\,. \tag{2.6.19}$$

Hint: *Formula (2.4.14).*

(b) Let ι_0 and ι_1 be the maps of U into $U \times A$ defined by $\iota_0(x) = (x,0)$ and $\iota_1(x) = (x,1)$. Show that (2.6.19) can be rewritten

$$\iota_0^*\omega - \iota_1^*\omega = dQ\alpha\,. \tag{2.6.20}$$

5. Let V be an open subset of $\mathbb{R}^m$ and $f_i : U \to V$, $i = 0, 1$, $\mathcal{C}^\infty$ maps. Suppose f_0 and f_1 are homotopic. Show that for every closed form, $\mu \in \Omega^k(V)$, $f_1^*\mu - f_0^*\mu$ is exact. *Hint:* Let $F : U \times A \to V$ be a

homotopy between f_0 and f_1 and let $\omega = F^*\mu$. Show that ω is closed and that $f_0^*\mu = \iota_0^*\omega$ and $f_1^*\mu = \iota_1^*\omega$. Conclude from (2.6.20) that

$$f_0^*\mu - f_1^*\mu = dQ\alpha \tag{2.6.21}$$

where $\omega = dt \wedge \alpha + \beta$ and α and β are reduced.

6. Show that if $U \subseteq \mathbb{R}^n$ is a contractable open set, then the Poincaré lemma holds: every closed form of degree $k > 0$ is exact.

7. An open subset, U, of $\mathbb{R}^n$ is said to be *star-shaped* if there exists a point $p_0 \in U$, with the property that for every point $p \in U$, the line segment,

$$tp + (1-t)p_0\,, \quad 0 \le t \le 1\,,$$

joining p to p_0 is contained in U. Show that if U is star-shaped it is contractable.

8. Show that the following open sets are star-shaped:

(a) The open unit ball

$$\{x \in \mathbb{R}^n\,,\ \|x\| < 1\}\,.$$

(b) The open rectangle, $I_1 \times \cdots \times I_n$, where each I_k is an open subinterval of $\mathbb{R}$.

(c) $\mathbb{R}^n$ itself.

(d) Product sets

$$U_1 \times U_2 \subseteq \mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$$

where U_i is a star-shaped open set in $\mathbb{R}^{n_i}$.

9. Let U be an open subset of $\mathbb{R}^n$, $f_t : U \to U$, $t \in \mathbb{R}$, a one-parameter group of diffeomorphisms and v its infinitesimal generator. Given $\omega \in \Omega^k(U)$ show that at $t = 0$

$$\frac{d}{dt} f_t^*\omega = L_v\omega\,. \tag{2.6.22}$$

Here is a sketch of a proof:

(a) Let $\gamma(t)$ be the curve, $\gamma(t) = f_t(p)$, and let φ be a zero-form, i.e., an element of $\mathcal{C}^\infty(U)$. Show that

$$f_t^* \varphi(p) = \varphi(\gamma(t))$$

and by differentiating this identity at $t = 0$ conclude that (2.5.40) holds for zero-forms.

(b) Show that if (2.5.40) holds for ω it holds for $d\omega$. *Hint:* Differentiate the identity

$$f_t^* \, d\omega = d f_t^* \omega$$

at $t = 0$.

(c) Show that if (2.5.40) holds for ω_1 and ω_2 it holds for $\omega_1 \wedge \omega_2$. *Hint:* Differentiate the identity

$$f_t^*(\omega_1 \wedge \omega_2) = f_t^* \omega_1 \wedge f_t^* \omega_2$$

at $t = 0$.

(d) Deduce (2.5.40) from a, b and c. *Hint:* Every k-form is a sum of wedge products of zero-forms and exact one-forms.

10. In exercise 9 show that for *all* t

$$\frac{d}{dt} f_t^* \omega = f_t^* L_v \omega = L_v f_t^* \omega \, . \tag{2.6.23}$$

Hint: By the definition of "one-parameter group", $f_{s+t} = f_s \circ f_t = f_r \circ f_s$, hence:

$$f_{s+t}^* \omega = f_t^*(f_s^* \omega) = f_s^*(f_t^* \omega) \, .$$

Prove the first assertion by differentiating the first of these identities with respect to s and then setting $s = 0$, and prove the second assertion by doing the same for the second of these identities.

In particular conclude that

$$f_t^* L_v \omega = L_v f_t^* \omega \, . \tag{2.6.24}$$

11. (a) By massaging the result above show that

$$\frac{d}{dt} f_t^* \omega \quad = \quad dQ_t \omega + Q_t \, d\omega \tag{2.6.25}$$

where

$$Q_t \omega \quad = \quad f_t^* \iota(v) \omega \, . \tag{2.6.26}$$

Hint: Formula (2.5.11).

(b) Let

$$Q\omega = \int_0^1 f_t^* \iota(v)\omega \, dt \, .$$

Prove the homotopy indentity

$$f_1^*\omega - f_0^*\omega = \, dQ\omega + Q\, d\omega \, . \tag{2.6.27}$$

12. Let U be an open subset of $\mathbb{R}^n$, V an open subset of $\mathbb{R}^m$, v a vector field on U, w a vector field on V and $f : U \to V$ a $\mathcal{C}^\infty$ map. Show that if v and w are f-related

$$\iota(v) f^* \omega = f^* \iota(w) \omega \, .$$

Hint: Chapter 1, §1.7, exercise 8.

2.7 Div, curl and grad

The basic operations in 3-dimensional vector calculus: grad, curl and div are, by definition, operations on *vector fields*. As we'll see below these operations are closely related to the operations

$$d : \Omega^k(\mathbb{R}^3) \to \Omega^{k+1}(\mathbb{R}^3) \tag{2.7.1}$$

in degrees $k = 0, 1, 2$. However, only two of these operations: grad and div, generalize to n dimensions. (They are essentially the d-operations in degrees zero and $n-1$.) And, unfortunately, there is no simple description in terms of vector fields for the other $n-2$ d-operations. This is one of the main reasons why an adequate theory of vector calculus in n-dimensions forces on one the differential form approach that we've developed in this chapter. Even in three dimensions, however, there is a good reason for replacing grad, div and curl by the three operations, (**??**). A problem that physicists spend a lot of time worrying about is the problem of *general covariance*: formulating the laws of physics in such a way that they admit as large a set of symmetries as possible, and frequently these formulations involve differential forms. An example is Maxwell's equations, the fundamental laws of electromagnetism. These are usually expressed as identities involving div and curl. However, as we'll explain below, there is an alternative formulation of Maxwell's equations based on the operations (**??**), and from the point of view of general covariance, this formulation is much more satisfactory: the only symmetries of $\mathbb{R}^3$ which preserve div and curl are translations and rotations, whereas the operations (2.7.1) admit all diffeomorphisms of $\mathbb{R}^3$ as symmetries.

To describe how grad, div and curl are related to the operations (**??**) we first note that there are two ways of converting vector fields into forms. The first makes use of the natural inner product, $B(v,w) = \sum v_i w_i$, on $\mathbb{R}^n$. From this inner product one gets by § **??**, exercise 9 a bijective linear map:

$$L : \mathbb{R}^n \to (\mathbb{R}^n)^* \tag{2.7.2}$$

with the defining property: $L(v) = \ell \Leftrightarrow \ell(w) = B(v,w)$. Via the identification (2.1.2) B and L can be transferred to $T_p\mathbb{R}^n$, giving one an inner product, B_p, on $T_p\mathbb{R}^n$ and a bijective linear map

$$L_p : T_p\mathbb{R}^n \to T_p^*\mathbb{R}^n \,. \tag{2.7.3}$$

Hence if we're given a vector field, $\mathfrak{v}$, on U we can convert it into a 1-form, $\mathfrak{v}^\sharp$, by setting

$$\mathfrak{v}^\sharp(p) = L_p\mathfrak{v}(p) \tag{2.7.4}$$

and this sets up a one–one correspondence between vector fields and 1-forms. For instance

$$\mathfrak{v} = \frac{\partial}{\partial x_i} \Leftrightarrow \mathfrak{v}^\sharp = dx_i \,, \tag{2.7.5}$$

(see exercise 3 below) and, more generally,

$$\mathfrak{v} = \sum f_i \frac{\partial}{\partial x_i} \Leftrightarrow \mathfrak{v}^\sharp = \sum f_i \, dx_i \,. \tag{2.7.6}$$

In particular if f is a $\mathcal{C}^\infty$ function on U the vector field "grad f" is by definition

$$\sum \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i} \tag{2.7.7}$$

and this gets converted by (2.7.8) into the 1-form, df. Thus the "grad" operation in vector calculus is basically just the operation, $d : \Omega^0(U) \to \Omega^1(U)$.

The second way of converting vector fields into forms is via the interior product operation. Namely let Ω be the n-form, $dx_1 \wedge \cdots \wedge dx_n$. Given an open subset, U of $\mathbb{R}^n$ and a $\mathcal{C}^\infty$ vector field,

$$v = \sum f_i \frac{\partial}{\partial x_i} \tag{2.7.8}$$

on U the interior product of v with Ω is the $(n-1)$-form

$$\iota(v)\Omega = \sum (-1)^{r-1} f_r dx_1 \wedge \cdots \wedge \widehat{dx}_r \cdots \wedge dx_n \,. \tag{2.7.9}$$

Moreover, every $(n-1)$-form can be written uniquely as such a sum, so (2.7.8) and (2.7.9) set up a one-one correspondence between vector fields and $(n-1)$-forms. Under this correspondence the d-operation gets converted into an operation on vector fields

$$v \to d\iota(v)\Omega \,. \tag{2.7.10}$$

Moreover, by (2.5.11)

$$d\iota(v)\Omega = L_v\Omega$$

and by (2.5.14)

$$L_v\Omega = \mathrm{div}(v)\Omega$$

where

$$\mathrm{div}(v) = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} \,. \tag{2.7.11}$$

In other words, this correspondence between $(n-1)$-forms and vector fields converts the d-operation into the divergence operation (2.7.11) on vector fields.

Notice that "div" and "grad" are well-defined as vector calculus operations in n-dimensions even though one usually thinks of them as operations in 3-dimensional vector calculus. The "curl" operation, however, is intrinsically a 3-dimensional vector calculus operation. To define it we note that by (2.7.9) every 2-form, μ, can be written uniquely as an interior product,

$$\mu = \iota(\mathfrak{w})\, dx_1 \wedge dx_2 \wedge dx_3 \,, \tag{2.7.12}$$

for some vector field $\mathfrak{w}$, and the left-hand side of this formula determines $\mathfrak{w}$ uniquely. Now let U be an open subset of $\mathbb{R}^3$ and $\mathfrak{v}$ a vector field on U. From $\mathfrak{v}$ we get by (2.7.6) a 1-form, $\mathfrak{v}^\sharp$, and hence by (2.7.12) a vector field, $\mathfrak{w}$, satisfying

$$d\mathfrak{v}^\sharp = \iota(\mathfrak{w})\, dx_1 \wedge dx_2 \wedge dx_3 \,. \tag{2.7.13}$$

The "curl" of $\mathfrak{v}$ is defined to be this vector field, in other words,

$$\operatorname{curl} \mathfrak{v} = \mathfrak{w} \,, \tag{2.7.14}$$

where $\mathfrak{v}$ and $\mathfrak{w}$ are related by (2.7.13).

We'll leave for you to check that this definition coincides with the definition one finds in calculus books. More explicitly we'll leave for you to check that if v is the vector field

$$v = f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2} + f_3 \frac{\partial}{\partial x_3} \tag{2.7.15}$$

then

$$\operatorname{curl} v = g_1 \frac{\partial}{\partial x_1} + g_2 \frac{\partial}{\partial x_2} + g_3 \frac{\partial}{\partial x_3} \tag{2.7.16}$$

where

$$\begin{aligned} g_1 &= \frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2} \\ g_2 &= \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} \\ g_3 &= \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} \,. \end{aligned} \tag{2.7.17}$$

To summarize: the grad, curl and div operations in 3-dimensions are basically just the three operations (**??**). The "grad" operation is the operation (**??**) in degree zero, "curl" is the operation (**??**) in degree one and "div" is the operation (**??**) in degree two. However, to define "grad" we had to assign an inner product, B_p, to the next tangent space, $T_p\mathbb{R}^n$, for each p in U; to define "div" we had to equip U with the 3-form, Ω, and to define "curl", the most complicated of these three operations, we needed the B_p's *and* Ω. This is why diffeomorphisms preserve the three operations (**??**) but don't preserve grad, curl and div. The additional structures which one needs to define grad, curl and div are only preserved by translations and rotations.

We will conclude this section by showing how Maxwell's equations, which are usually formulated in terms of div and curl, can be reset into "form" language. (The paragraph below is an abbreviated version of Guillemin–Sternberg, *Symplectic Techniques in Physics*, §1.20.)

Maxwell's equations assert:

$$\operatorname{div} \mathfrak{v}_E = q \tag{2.7.18}$$

$$\operatorname{curl} \mathfrak{v}_E = -\frac{\partial}{\partial t}\mathfrak{v}_M \tag{2.7.19}$$

$$\operatorname{div} \mathfrak{v}_M = 0 \tag{2.7.20}$$

$$c^2 \operatorname{curl} \mathfrak{v}_M = \mathfrak{w} + \frac{\partial}{\partial t}\mathfrak{v}_E \tag{2.7.21}$$

where $\mathfrak{v}_E$ and $\mathfrak{v}_M$ are the *electric* and *magnetic* fields, q is the *scalar charge density*, $\mathfrak{w}$ is the *current density* and c is the velocity of light. (To simplify (2.7.25) slightly we'll assume that our units of space–time are chosen so that $c = 1$.) As above let $\Omega = dx_1 \wedge dx_2 \wedge dx_3$ and let

$$\mu_E = \iota(\mathfrak{v}_E)\Omega \tag{2.7.22}$$

and

$$\mu_M = \iota(\mathfrak{v}_M)\Omega\,. \tag{2.7.23}$$

We can then rewrite equations (2.7.18) and (2.7.20) in the form

$$d\mu_E = q\Omega \tag{2.7.18$'$}$$

and

$$d\mu_M = 0\,. \tag{2.7.20$'$}$$

What about (2.7.19) and (2.7.21)? We will leave the following "form" versions of these equations as an exercise.

$$d\mathfrak{v}_E^\sharp = -\frac{\partial}{\partial t}\mu_M \tag{2.7.19$'$}$$

and

$$d\mathfrak{v}_M^\sharp = \iota(\mathfrak{w})\Omega + \frac{\partial}{\partial t}\mu_E \tag{2.7.21$'$}$$

where the 1-forms, $\mathfrak{v}_E^\sharp$ and $\mathfrak{v}_M^\sharp$, are obtained from $\mathfrak{v}_E$ and $\mathfrak{v}_M$ by the operation, (2.7.4).

These equations can be written more compactly as differential form identities in $3+1$ dimensions. Let ω_M and ω_E be the 2-forms

$$\omega_M = \mu_M - \mathfrak{v}_E^\sharp \wedge dt \tag{2.7.24}$$

and

$$\omega_E = \mu_E - \mathfrak{v}_M^\sharp \wedge dt \tag{2.7.25}$$

and let Λ be the 3-form

$$\Lambda = q\Omega + \iota(\mathfrak{w})\Omega \wedge dt\,. \tag{2.7.26}$$

We will leave for you to show that the four equations (2.7.18) — (2.7.21) are equivalent to two elegant and compact (3+1)-dimensional identities

$$d\omega_M = 0 \tag{2.7.27}$$

and

$$d\omega_E = \Lambda\,. \tag{2.7.28}$$

Exercises.

1. Verify that the "curl" operation is given in coordinates by the formula (2.7.17).

2. Verify that the Maxwell's equations, (2.7.18) and (2.7.19) become the equations (2.7.20) and (2.7.21) when rewritten in differential form notation.

3. Show that in $(3+1)$-dimensions Maxwell's equations take the form (2.7.17)–(2.7.18).

4. Let U be an open subset of $\mathbb{R}^3$ and v a vector field on U. Show that if v is the gradient of a function, its curl has to be zero.

5. If U is simply connected prove the converse: If the curl of v vanishes, v is the gradient of a function.

6. Let $w = \operatorname{curl} v$. Show that the divergence of w is zero.

7. Is the converse statment true? Suppose the divergence of w is zero. Is $w = \operatorname{curl} v$ for some vector field v?

2.8 Symplectic geometry and classical mechanics

In this section we'll describe some other applications of the theory of differential forms to physics. Before describing these applications, however, we'll say a few words about the geometric ideas that are involved. Let $x_1, \ldots, x_{2n}$ be the standard coordinate functions on $\mathbb{R}^{2n}$ and for $i = 1, \ldots, n$ let $y_i = x_{i+n}$. The two-form

$$\omega = \sum_{i=1}^{n} dx_i \wedge jy_i \tag{2.8.1}$$

is known as the *Darboux* form. From the identity

$$\omega = -d\left(\sum y_i \, dx_i\right) . \tag{2.8.2}$$

it follows that ω is exact. Moreover computing the n-fold wedge product of ω with itself we get

$$\begin{aligned} \omega^n &= \left(\sum_{i_i=1}^{n} dx_{i_1} \wedge dy_{i_1}\right) \wedge \cdots \wedge \left(\sum_{i_n=1}^{n} dx_{i_n} \wedge dy_{i_n}\right) \\ &= \sum_{i_1,\ldots,i_n} dx_{i_1} \wedge dy_{i_1} \wedge \cdots \wedge dx_{i_n} \wedge dy_{i_n} . \end{aligned}$$

We can simplify this sum by noting that if the multi-index, $I = i_1, \ldots, i_n$, is repeating the wedge product

$$dx_{i_1} \wedge dy_{i_1} \wedge \cdots \wedge dx_{i_n} \wedge dx_{i_n} \tag{2.8.3}$$

involves two repeating dx_{i_1}'s and hence is zero, and if I is non-repeating we can permute the factors and rewrite (2.8.3) in the form

$$dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n .$$

(See §1.6, exercise 5.) Hence since these are exactly $n!$ non-repeating multi-indices

$$\omega^n = n! \, dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$$

i.e.,

$$\frac{1}{n!}\omega^n = \Omega \tag{2.8.4}$$

where

$$\Omega = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \tag{2.8.5}$$

is the *symplectic volume form* on $\mathbb{R}^{2n}$.

Let U and V be open subsets of $\mathbb{R}^{2n}$. A diffeomorphism $f : U \to V$ is said to be a *symplectic* diffeomorphism (or *symplectomorphism* for short) if $f^*\omega = \omega$. In particular let

$$f_t : U \to U\,, \quad -\infty < t < \infty \tag{2.8.6}$$

be a one-parameter group of diffeomorphisms and let v be the vector field generating (2.8.6). We will say that v is a *symplectic* vector field if the diffeomorphisms, (2.8.6) are symplectomorphisms, i.e., for all t,

$$f_t^*\omega = \omega\,. \tag{2.8.7}$$

Let's see what such vector fields have to look like. Note that by (2.6.23)

$$\frac{d}{dt} f_t^*\omega = f_t^* L_v\omega\,, \tag{2.8.8}$$

hence if $f_t^*\omega = \omega$ for all t, the left hand side of (2.8.8) is zero, so

$$f_t^* L_v\omega = 0\,.$$

In particular, for $t = 0$, f_t is the identity map so $f_t^* L_v\omega = L_v\omega = 0$. Conversely, if $L_v\omega = 0$, then $f_t^* L_v\omega = 0$ so by (2.8.8) $f_t^*\omega$ doesn't depend on t. However, since $f_t^*\omega = \omega$ for $t = 0$ we conclude that $f_t^*\omega = \omega$ for all t. Thus to summarize we've proved

Theorem 2.8.1. *Let $f_t : U \to U$ be a one-parameter group of diffeomorphisms and v the infinitesmal generator of this group. Then v is symplectic of and only if $L_v\omega = 0$.*

There is an equivalent formulation of this result in terms of the interior product, $\iota(v)\omega$. By (2.5.11)

$$L_v\omega = d\iota(v)\omega + \iota(v)\,d\omega\,.$$

But by (2.8.2) $d\omega = 0$ so

$$L_v\omega = d\iota(v)\omega\,.$$

Thus we've shown

Theorem 2.8.2. *The vector field v is symplectic if and only if $\iota(v)\omega$ is closed.*

If $\iota(v)\omega$ is not only closed but is exact we'll say that v is a *Hamiltonian* vector field. In other words v is Hamiltonian if

$$\iota(v)\omega = dH \tag{2.8.9}$$

for some $\mathcal{C}^\infty$ functions, $H \in \mathcal{C}^\infty(U)$.

Let's see what this condition looks like in coordinates. Let

$$v = \sum f_i \frac{\partial}{\partial x_i} + g_i \frac{\partial}{\partial y_i}\,. \tag{2.8.10}$$

Then

$$\begin{aligned}\iota(v)\omega &= \sum_{i,j} f_i \iota\left(\frac{\partial}{\partial x_i}\right) dx_j \wedge dy_j \\ &\quad + \sum_{i,j} g_i \iota\left(\frac{\partial}{\partial y_i}\right) dx_j \wedge dy_i\,.\end{aligned}$$

But

$$\iota\left(\frac{\partial}{\partial x_i}\right) dx_j = \begin{cases} 1 & i = i \\ 0 & i \neq j \end{cases}$$

and

$$\iota\left(\frac{\partial}{\partial x_i}\right) dy_j = 0$$

so the first summand above is

$$\sum f_i\, dy_i$$

and a similar argument shows that the second summand is

$$-\sum g_i\, dx_i\,.$$

Hence if v is the vector field (2.8.10)

$$\iota(v)\omega = \sum f_i\, dy_i - g_i\, dx_i\,. \tag{2.8.11}$$

Thus since

$$dH = \sum \frac{\partial H}{\partial x_i}\, dx_i + \frac{\partial H}{\partial y_i}\, dy_i$$

we get from (2.8.9)–(2.8.11)

$$f_i = \frac{\partial H}{\partial y_i} \text{ and } g_i = -\frac{\partial H}{\partial x_i} \tag{2.8.12}$$

so v has the form:

$$v = \sum \frac{\partial H}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial y_i}\,. \tag{2.8.13}$$

In particular if $\gamma(t) = (x(t)\,,\, y(t))$ is an integral curve of v it has to satisfy the system of differential equations

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{\partial H}{\partial y_i}(x(t)\,,\, y(t)) \\ \frac{dy_i}{dt} &= -\frac{\partial H}{\partial x_i}(x(t)\,,\, y(t))\,. \end{aligned} \tag{2.8.14}$$

The formulas (2.8.10) and (2.8.11) exhibit an important property of the Darboux form, ω. Every one-form on U can be written uniquely as a sum

$$\sum f_i\, dy_i - g_i\, dx_i$$

with f_i and g_i in $\mathcal{C}^\infty(U)$ and hence (2.8.10) and (2.8.11) imply

Theorem 2.8.3. *The map, $v \to \iota(v)\omega$, sets up a* one-one correspondence *between vector field and one-forms.*

In particular for every $\mathcal{C}^\infty$ function, H, we get by correspondence a unique vector field, $v = v_H$, with the property (2.8.9).

We next note that by (**??**)

$$L_v H = \iota(v)\, dH = \iota(v)(\iota(v)\omega) = 0\,.$$

Thus

$$L_v H = 0 \tag{2.8.15}$$

i.e., H is an integral of motion of the vector field, v. In particular if the function, $H : U \to \mathbb{R}$, is proper, then by Theorem **??** the vector field, v, is complete and hence by Theorem 2.8.1 generates a one-parameter group of symplectomorphisms.

One last comment before we discuss the applications of these results to classical mechanics. If the one-parameter group (2.8.6) is a group of symplectomorphisms then $f_t^*\omega^n = f_t^*\omega \wedge \cdots \wedge f_t^*\omega = \omega^n$ so by (2.8.4)

$$f_t^*\Omega = \Omega \tag{2.8.16}$$

where Ω is the symplectic volume form (2.8.5).

The application we want to make of these ideas concerns the description, in Newtonian mechanics, of a physical system consisting of N interacting point-masses. The *configuration space* of such a system is

$$\mathbb{R}^n = \mathbb{R}^3 \times \cdots \times \mathbb{R}^3 \qquad (N \text{ copies})$$

with position coordinates, $x_1, \ldots, x_n$ and the *phase space* is $\mathbb{R}^{2n}$ with position coordinates $x_1, \ldots, x_n$ and momentum coordinates, $y_1, \ldots, y_n$. The *kinetic energy* of this system is a quadratic function of the momentum coordinates

$$\frac{1}{2}\sum \frac{1}{m_i} y_i^2\,, \tag{2.8.17}$$

and for simplicity we'll assume that the potential energy is a function, $V(x_1, \ldots, x_n)$, of the position coordinates alone, i.e., it doesn't depend on the momenta and is time-independent as well. Let

$$H = \frac{1}{2}\sum \frac{1}{m_i} y_i^2 + V(x_1, \ldots, x_n) \tag{2.8.18}$$

be the *total energy* of the system. We'll show below that Newton's second law of motion in classical mechanics reduces to the assertion: *the trajectories in phase space of the system above are just the integral curves of the Hamiltonian vector field,* v_H.

Proof. For the function (2.8.18) the equations (2.8.14) become

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{1}{m_i} y_i \\ \frac{dy_i}{dt} &= -\frac{\partial V}{\partial x_i}\,. \end{aligned} \tag{2.8.19}$$

The first set of equation are essentially just the definitions of momenta, however, if we plug them into the second set of equations we get

$$m_i \frac{d^2 x_i}{dt^2} = -\frac{\partial V}{\partial x_i} \tag{2.8.20}$$

and interpreting the term on the right as the force exerted on the i^{th} point-mass and the term on the left as mass times acceleration this equation becomes Newton's second law.

□

In classical mechanics the equations (2.8.14) are known as the Hamilton–Jacobi equations. For a more detailed account of their role in classical mechanics we highly recommend Arnold's book, *Mathematical Methods of Classical Mechanics*. Historically these equations came up for the first time, not in Newtonian mechanics, but in gemometric optics and a brief description of their origins there and of their relation to Maxwell's equations can be found in the bookl we cited above, *Symplectic Techniques in Physics*.

We'll conclude this chapter by mentioning a few implications of the Hamiltonian description (2.8.14) of Newton's equations (2.8.20).

1. *Conservation of energy.* By (2.8.15) the energy function (2.8.18) is constant along the integral curves of v, hence the energy of the system (2.8.14) doesn't change in time.

2. *Noether's principle.* Let $\gamma_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a one-parameter group of diffeomorphisms of phase space and w its infinitesmal generator. The γ_t's are called a *symmetry* of the system above if

(a) They preserve the function (2.8.18)
and
(b) the vector field w is Hamiltonian.

The condition (b) means that

$$\iota(w)\omega = dG \tag{2.8.21}$$

for some $\mathcal{C}^\infty$ function, G, and what Noether's principle asserts is that *this function is an integral of motion of the system* (2.8.14), i.e., satisfies $L_v G = 0$. In other words stated more succinctly: symmetries of the system (2.8.14) give rise to integrals of motion.

3. *Poincaré recurrence.* An important theorem of Poincaré asserts that if the function $H : \mathbb{R}^{2n} \to \mathbb{R}$ defined by (2.8.18) is proper then every trajectory of the system (2.8.14) returns arbitrarily close to its initial position at some positive time, t_0, and, in fact, does this not just once but does so infinitely often. We'll sketch a proof of this theorem, using (2.8.16), in the next chapter.

Exercises.

1. Let v_H be the vector field (2.8.13). Prove that $\operatorname{div}(v_H) = 0$.

2. Let U be an open subset of $\mathbb{R}^m$, $f_t : U \to U$ a one-parameter group of diffeomorphisms of U and v the infinitesmal generator of this group. Show that if α is a k-form on U then $f_t^*\alpha = \alpha$ for all t if and only if $L_v\alpha = 0$ (i.e., generalize to arbitrary k-forms the result we proved above for the Darboux form).

3. The harmonic oscillator. Let H be the function $\sum_{i=1}^n m_i(x_i^2 + y_i^2)$ where the m_i's are positive constants.

(a) Compute the integral curves of v_H.

(b) Poincaré recurrence. Show that if $(x(t), y(t))$ is an integral curve with initial point $(x_0, y_0) = (x(0), y(0))$ and U an arbitrarily small neighborhood of (x_0, y_0), then for *every* $c > 0$ there exists a $t > c$ such that $(x(t), y(t)) \in U$.

4. Let U be an open subset of $\mathbb{R}^{2n}$ and let H_i, $i = 1, 2$, be in $\mathcal{C}^\infty(U)_i$. Show that

$$[v_{H_1}, v_{H_2}] = v_H \tag{2.8.22}$$

where

$$H = \sum_{i=1}^n \frac{\partial H_1}{\partial x_i}\frac{\partial H_2}{\partial y_i} - \frac{\partial H_2}{\partial x_i}\frac{\partial H_1}{\partial y_i}\,. \tag{2.8.23}$$

5. The expression (2.8.23) is known as the *Poisson bracket* of H_1 and H_2 and is denoted by $\{H_1, H_2\}$. Show that it is anti-symmetric

$$\{H_1, H_2\} = -\{H_2, H_1\}$$

and satisfies Jacobi's identity

$$0 = \{H_1, \{H_2, H_3\}\} + \{H_2, \{H_3, H_1\}\} + \{H_3, \{H_1, H_2\}\} .$$

6. Show that

$$\{H_1, H_2\} = L_{v_{H_1}} H_2 = -L_{v_{H_2}} H_1 . \tag{2.8.24}$$

7. Prove that the following three properties are equivalent.

(a) $\{H_1, H_2\} = 0$.

(b) H_1 is an integral of motion of v_2.

(c) H_2 is an integral of motion of v_1.

8. Verify Noether's principle.

9. Conservation of linear momentum. Suppose the potential, V in (2.8.18) is invariant under the one-parameter group of translations

$$T_t(x_1, \dots, x_n) = (x_1 + t, \dots, x_n + t) .$$

(a) Show that the function (2.8.18) is invariant under the group of diffeomorphisms

$$\gamma_t(x, y) = (T_t x, y) .$$

(b) Show that the infinitesmal generator of this group is the Hamiltonian vector field v_G where $G = \sum_{i=1}^n y_i$.

(c) Conclude from Noether's principle that this function is an integral of the vector field v_H, i.e., that "total linear moment" is conserved.

(d) Show that "total linear momentum" is conserved if V is the Coulomb potential

$$\sum_{i \neq j} \frac{m_i}{|x_i - x_j|} .$$

10. Let $R_t^i : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the rotation which fixes the variables, (x_k, y_k), $k \neq i$ and rotates (x_i, y_i) by the angle, t:

$$R_t^i(x_i, y_i) = (\cos t\, x_i + \sin t\, y_i \, , \, -\sin t\, x_i + \cos t\, y_i) .$$

(a) Show that R_t^i, $-\infty < t < \infty$, is a one-parameter group of symplectomorphisms.

(b) Show that its generator is the Hamiltonian vector field, v_{H_i}, where $H_i = (x_i^2 + y_i^2)/2$.

(c) Let H be the "harmonic oscillator" Hamiltonian in exercise 3. Show that the R_t^j's preserve H.

(d) What does Noether's principle tell one about the classical mechanical system with energy function H?

11. Show that if U is an open subset of $\mathbb{R}^{2n}$ and v is a symplectic vector field on U then for every point, $p_0 \in U$, there exists a neighborhood, U_0, of p_0 on which v is Hamiltonian.

12. Deduce from exercises 4 and 11 that if v_1 and v_2 are symplectic vector fields on an open subset, U, of $\mathbb{R}^{2n}$ their Lie bracket, $[v_1, v_2]$, is a Hamiltonian vector field.

13. Let α be the one-form, $\sum_{i=1}^n y_i \, dx_i$.

(a) Show that $\omega = -\, d\alpha$.

(b) Show that if α_1 is any one-form on $\mathbb{R}^{2n}$ with the property, $\omega = -d\alpha_1$, then

$$\alpha = \alpha_1 + F$$

for some $\mathcal{C}^\infty$ function F.

(c) Show that $\alpha = \iota(w)\omega$ where w is the vector field

$$-\sum y_i \frac{\partial}{\partial y_i} .$$

14. Let U be an open subset of $\mathbb{R}^{2n}$ and v a vector field on U. Show that v has the property, $L_v \alpha = 0$, if and only if

$$\iota(v)\omega = d\iota(v)\alpha \, . \tag{2.8.25}$$

In particular conclude that if $L_v \alpha = 0$ then v is Hamiltonian. *Hint:* (2.8.2).

15. Let H be the function

$$H(x, y) = \sum f_i(x) y_i \, , \tag{2.8.26}$$

where the f_i's are $\mathcal{C}^\infty$ functions on $\mathbb{R}^n$. Show that

$$L_{v_H} \alpha = 0 \, . \tag{2.8.27}$$

16. Conversely show that if H is any $\mathcal{C}^\infty$ function on $\mathbb{R}^{2n}$ satisfying (2.8.27) it has to be a function of the form (2.8.26). *Hints:*

(a) Let v be a vector field on $\mathbb{R}^{2n}$ satisfying $L_v\alpha = 0$. By the previous exercise $v = v_H$, where $H = \iota(v)\alpha$.

(b) Show that H has to satisfy the equation

$$\sum_{i=1}^n y_i \frac{\partial H}{\partial y_i} = H\,.$$

(c) Conclude that if $H_r = \frac{\partial H}{\partial y_r}$ then H_r has to satisfy the equation

$$\sum_{i=1}^n y_i \frac{\partial}{\partial y_i} H_r = 0\,.$$

(d) Conclude that H_r has to be constant along the rays (x, ty), $0 \le t < \infty$.

(e) Conclude finally that H_r has to be a function of x alone, i.e., doesn't depend on y.

17. Show that if $v_{\mathbb{R}^n}$ is a vector field

$$\sum f_i(x) \frac{\partial}{\partial x_i}$$

on configuration space there is a unique lift of $v_{\mathbb{R}^n}$ to phase space

$$v = \sum f_i(x) \frac{\partial}{\partial x_i} + g_i(x,y) \frac{\partial}{\partial y_i}$$

satisfying $L_v\alpha = 0$.